ELE539A: Optimization of Communication Systems
Lecture 13: Semidefinite Programming and Estimation Applications

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Lecture Outline

• Cones and dual cones
• Generalized inequality and convexity
• Conic programming
• Semidefinite programming (SDP)
• Application: Multi-user detection

Thanks: Stephen Boyd (Some materials from Boyd and Vandenberghe)
Cones and Convex Cones

$C$ is a cone if for every $x \in C$ and $\theta \geq 0$, we have $\theta x \in C$

$C$ is a convex cone if it is convex and a cone: for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$

$$\theta_1 x_1 + \theta_2 x_2 \in C$$
Norm Cones

Given a norm, norm cone is a convex cone:

\[ C = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\} \]

Example: second order cone:

\[
C = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\} \\
= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \mid \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \leq 0, t \geq 0 \right\}
\]
Positive Semidefinite Cone

Matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite $A \succeq 0$ if for all $x \in \mathbb{R}^n$, 

$$x^T Ax \geq 0$$

Matrix $A \in \mathbb{R}^{n \times n}$ is positive definite $A \succ 0$ if for all $x \in \mathbb{R}^n$, 

$$x^T Ax > 0$$

Set of symmetric positive semidefinite matrices:

$$S_+^n = \{ X \in \mathbb{R}^{n \times n} | X = X^T, X \succeq 0 \}$$

$S_+^n$ is a convex cone: if $\theta_1, \theta_2 \geq 0$ and $A, B \in S_+^n$, then $\theta_1 A + \theta_2 B \in S_+^n$, since for all $x \in \mathbb{R}^n$:

$$x^T (\theta_1 A + \theta_2 B)x = \theta_1 x^T Ax + \theta_2 x^T Bx \geq 0$$
Proper Cones and Generalized Inequalities

A cone $K$ is a proper cone if

- $K$ is convex
- $K$ is closed
- $K$ has nonempty interior
- $K$ has no lines ($x \in K, -x \in K \Rightarrow x = 0$)

Proper cone $K$ induces a generalized inequality (partial ordering on $\mathbb{R}^n$):

\[
\begin{align*}
x \preceq_K y & \iff y - x \in K \\
x \prec_K y & \iff y - x \in \text{int } K
\end{align*}
\]
Examples

• **Nonnegative orthant** and **componentwise inequality**:

  \( K = \mathbb{R}_+^n \) is a proper cone

  \( x \preceq_K y \) means \( x_i \leq y_i, \ i = 1, \ldots, n \)

  \( x \prec_K y \) means \( x_i < y_i, \ i = 1, \ldots, n \)

• **Positive semidefinite cone** and **matrix inequality**:

  \( K = S_+^n \) is a proper cone in the set of symmetric matrices \( S^n \)

  \( X \preceq_K Y \) means \( Y - X \) is positive semidefinite

  \( X \prec_K Y \) means \( Y - X \) is positive definite.
Properties of Generalized Inequalities

- If \( x \preceq_K y \) and \( u \preceq_K v \), then \( x + u \preceq_K y + v \)
- If \( x \preceq_K y \) and \( y \preceq_K z \), then \( x \preceq_K z \)
- If \( x \preceq_K y \) and \( \alpha \geq 0 \), then \( \alpha x \preceq_K \alpha y \)
- If \( x \preceq_K y \) and \( y \preceq_K x \), then \( x = y \)
- If \( x_i \preceq_K y_i \) for \( i = 1, \ldots \), and \( x_i \to x \) and \( y_i \to y \) as \( i \to \infty \), then \( x \preceq_K y \)
Dual Cones

Given a cone $K$. **Dual cone of $K$**:

$$K^* = \{ y | x^T y \geq 0 \ \forall x \in K \}$$

$K^*$ is always a convex cone

$K$ is proper cone $\Rightarrow K^*$ is a proper cone and $K^{**} = K$

- Nonnegative orthant cone is self-dual
Dual PSD Cone

Consider inner product $X^T Y = \text{tr}(XY) = \sum_{i,j} X_{ij} Y_{ij}$ on $\mathbb{S}^n$. PSD cone $\mathbb{S}_+^n$ is self-dual:

$$\text{tr}(XY) \geq 0 \quad \forall X \succeq 0 \iff Y \succeq 0$$

Proof: Forward direction: suppose $Y \not\succeq 0$. Then $\exists q \in \mathbb{R}^n$ such that $q^T Y q = \text{tr}(qq^T Y) < 0$. Therefore, $X = qq^T$ satisfies $\text{tr}(XY) < 0$.

Reverse direction: suppose $X, Y \succeq 0$. Express $X$ by eigenvalue decomposition: $X = \sum_{i=1}^n \lambda_i q_i q_i^T$ where $\lambda_i \geq 0$, $i = 1, \ldots, n$. Then

$$\text{tr}XY = \text{tr} \left( Y \sum_{i=1}^n \lambda_i q_i q_i^T \right) = \sum_{i=1}^n \lambda_i q_i^T Y q_i \geq 0$$
Dual Generalized Inequality

Given proper cone $K$ and generalized inequality $\succeq_K$. Dual cone $K^*$ is also proper and induces dual generalized inequality $\succeq_{K^*}$

Relationship between $\succeq_K$ and $\succeq_{K^*}$:

1. $x \succeq_K y$ iff $\lambda^T x \leq \lambda^T y$ for all $\lambda \succeq_{K^*} 0$

2. $x \prec_K y$ iff $\lambda^T x < \lambda^T y$ for all $\lambda \succeq_{K^*} 0, \lambda \neq 0$

Since $K^{**} = K$, these properties hold if $\succeq_K$ and $\succeq_{K^*}$ are interchanged
Generalized Inequality Induced Monotonicity

$f : \mathbb{R}^n \to \mathbb{R}$ is $K$-nondecreasing (increasing) if

$$x \preceq_K y \Rightarrow f(x) \leq (\prec) f(y)$$

First order condition for differentiable $f$ with convex domain: $f$ is $K$-nondecreasing iff:

$$\nabla f(x) \succeq_K 0$$

$f$ is $K$-increasing if:

$$\nabla f(x) \succ_K 0$$

Example: For PSD cone, $\text{tr}(X^{-1})$ is matrix decreasing on $S_{++}^n$ and $\det X$ is matrix increasing on $S_+^n$
Generalized Inequality Induced Convexity

$f : \mathbb{R}^n \to \mathbb{R}^m$ is $K$-convex if for all $x, y$ and $\theta \in [0, 1]$,

$$f(\theta x + (1 - \theta)y) \preceq_K \theta f(x) + (1 - \theta)f(y)$$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is strictly $K$-convex if for all $x \neq y$ and $\theta \in (0, 1)$,

$$f(\theta x + (1 - \theta)y) \prec_K \theta f(x) + (1 - \theta)f(y)$$

First order condition: For differentiable $f$ with convex domain, $f$ is $K$-convex iff for all $x, y \in \text{dom} f$,

$$f(y) \succeq_K f(x) + Df(x)(y - x)$$
Matrix Convexity

$f$ is a symmetric-matrix-valued function. $f : \mathbb{R}^{n \times m} \rightarrow \mathbb{S}^m$. $f$ is convex with respect to matrix inequality if for any $x, y \in \text{dom } f$ and $\theta \in [0, 1]$, 

$$f(\theta x + (1 - \theta)y) \preceq \theta f(x) + (1 - \theta)f(y)$$

Equivalently, $f$ is matrix convex iff scaler-valued function $z^T f(x) z$ is convex for all $z$

- $f(X) = XX^T$ is matrix convex
- $f(X) = X^p$ is matrix convex for $p \in [1, 2]$ or $p \in [-1, 0]$, and matrix concave for $p \in [0, 1]$
- $f(X) = e^X$ is not matrix convex (unless $X$ is a scalar)
**Generalized Inequality Constraints**

Convex optimization with generalized inequality constraints on vector-valued functions:

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \preceq_{K_i} 0, \quad i = 1, 2, \ldots, m
\end{align*}
\]

\(f_0 : \mathbb{R}^n \to \mathbb{R}, \quad f_i : \mathbb{R}^n \to \mathbb{R}^{k_i}\) are \(K_i\)-convex for some proper cones \(K_i\)

- Feasible set is convex
- Local optimality \(\Rightarrow\) global optimality
Conic Programming

Linear programming with linear generalized inequality constraint:

minimize \( c^T x \)

subject to \( Fx + G \preceq_K 0 \)
\( Ax = b \)

- When \( K \) is nonnegative orthant, conic program reduces to LP
- When \( K \) is PSD cone, write inequality constraints as Linear Matrix Inequalities (LMI):
  \( x_1 F_1 + \ldots + x_n F_n + G \preceq 0 \)

where \( F_i, G \in S^k \). When they are diagonal, LMI reduces to linear inequalities
**SDP**

**SDP**: Minimize linear objective over linear equalities and LMI on variables \( x \in \mathbb{R}^n \)

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad x_1 F_1 + \ldots + x_n F_n + G \preceq 0 \\
& \quad Ax = b
\end{align*}
\]

**SDP in standard form**: Minimize a matrix inner product over equality constraints on inner products on variables \( X \in \mathbb{S}^n \)

\[
\begin{align*}
\text{minimize} & \quad \text{tr}(CX) \\
\text{subject to} & \quad \text{tr}(A_i X) = b_i, \quad i = 1, 2, \ldots, p \\
& \quad X \succeq 0
\end{align*}
\]
LP and SOCP as SDP

LP as SDP:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \text{diag}(Gx - h) \preceq 0 \\
& \quad Ax = b
\end{align*}
\]

SOCP:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, N \\
& \quad Fx = g
\end{align*}
\]

SOCP as SDP:

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \begin{bmatrix}
(c_i x + d_i)I & A_i x + b_i \\
(A_i x + b_i)^T & (c_i x + d_i)I
\end{bmatrix} \succeq 0, \quad i = 1, \ldots, N \\
& \quad Fx = g
\end{align*}
\]
Matrix Norm Minimization

\[ A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n \] where \( A_i \in \mathbb{R}^{p \times q} \). Consider unconstrained spectral norm (max. singular value) minimization over \( x \):

\[
\text{minimize} \| A(x) \|_2
\]

which is equivalent to convex optimization with LMI on \((x, t)\):

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad A(x)^T A(x) \preceq t^2 I
\end{align*}
\]

which is equivalent to SDP

\[
\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \preceq 0
\end{align*}
\]
Related Problems

SDP problems:
- Minimize largest eigenvalue
- Minimize sum of $r$ largest eigenvalues
- Maximize log determinant

SDP approximations:
- Combinatorial optimization
- Rank minimization
Multi-user Detection

Received signal of $K$-user basic synchronous CDMA channel:

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + n(t), \quad t \in [0, T]$$

Amplitude $A_k$, signature waveform $s_k(t)$, information bit $b_k \in \{-1, +1\}$, noise $n(t)$, period $T$

ML detection: find $b$ that minimizes

$$\int_0^T \left[ y(t) - \sum_{k=1}^{K} A_k b_k s_k(t) \right]^2 dt$$

$$= \int_0^T \left[ \sum_{k=1}^{K} A_k b_k s_k(t) \right] dt - 2 \int_0^T \left[ \sum_{k=1}^{K} A_k b_k s_k(t) \right] y(t) dt$$

$$= b^T H b - 2 b^T A y$$
Boolean Constrained QP Formulation

\[ A = \text{diag}(A) : \text{amplitude matrix} \]

\[ R_{ij} = \int_0^T s_i(t)s_j(t)dt : \text{cross-correlation matrix} \]

\[ y = RA_b + n : \text{sampled matched filter output} \]

Notation: \( H = ARA \) and \( p = -2Ay \)

- minimize \( x^T Hx + p^T x \)
- subject to \( x_i \in \{-1, +1\}, \ i = 1, \ldots, K \)

Equivalent form: let \( X = xx^T \)

- subject to \( X_{ii} = 1, \ i = 1, \ldots, K, \)
- \( X \succeq 0, \ \text{rank}(X) = 1 \)
SDP Relaxation

Neglect rank constraint:

subject to \( X_{ii} = 1, \ i = 1, \ldots, K, \)
\( X \succeq 0 \)

Let \( \hat{X} = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix} \) and \( C = \begin{bmatrix} H & p/2 \\ p^T/2 & 1 \end{bmatrix} \), reformulate as SDP:

minimize \( \text{tr}(C\hat{X}) \)
subject to \( \hat{X}_{ii} = 1, \ i = 1, \ldots, K + 1, \)
\( \hat{X} \succeq 0 \)

Factorization and randomization methods to recover solution to original boolean constrained QP from SDP relaxation solutions
Dual Relaxation

Relax $x^T x = K$ to $x^T x \leq K$:

\[
\begin{align*}
\text{minimize} & \quad x^T H x + p^T x \\
\text{subject to} & \quad x^T x \leq K
\end{align*}
\]

Lagrange dual problem:

\[
\begin{align*}
\text{minimize} & \quad -\frac{1}{4} p^T (H + \lambda I)^{-1} p - \lambda K \\
\text{subject to} & \quad \lambda \geq 0
\end{align*}
\]

Gradient-descent method solves this convex optimization in one variable $\lambda$

Recover primal optimal solution: $x^* = (H + \lambda^* I)^{-1} Ay$
**Complexity Performance Tradeoff**

Exhaustive search: \( \text{NP-hard } O(2^K) \)

Successive interference cancellation

Relaxation methods:
- SDP relaxation: polynomial time \( O(K^3) \) (SDP solution dominates randomization in terms of computational load)
- Unconstrained relaxation: analytic solution (related to decorrelator and linear MMSE)
- Bound relaxation
- Dual relaxation

Better complexity-tradeoff possible through duality, randomization, and other relaxation methods?
Lecture Summary

• Proper cones induce generalized inequalities in $\mathbb{R}^n$ and $\mathbb{S}^n$, which induces generalized convex inequality constraints

• Convex optimization with generalized inequalities: conic programming

• SDP is a conic programming over PSD cone with LMI, includes LP, QP, QCQP, SOCP as special cases

• Relaxation for multi-user detection problems

Reading: Sections 2.1, 2.6, 3.6, 4.6, 5.9 in Boyd and Vandenberghe.
