Problem Set #3

1. *Energy (4 pts).*
What are the energies of these signals (where t is the independent variable)?

   (a) $x(t) = Ae^{-at}u(t)$ with $a > 0$ (Note: $u(t)$ is the unit step function defined as $1_{[0,\infty)}(t)$).
   (b) The unit area rectangular pulse of width a, $\Delta_a(t)$.

2. *Power (4 pts).*
What are the powers of these signals?

   (a) $x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$.
   (b) $x(t) = \sum_{k=-N}^{N} A_k e^{i\omega_0 kt}$.

3. *Orthogonality and orthonormality (9 pts).*
Two functions are said to be *orthogonal* if

$$\int_{-\infty}^{\infty} f(x)\, g^*(x)\, \mathrm{d}x = 0.$$

In addition to orthogonality, if both $\int_{-\infty}^{\infty} |f(x)|^2\, \mathrm{d}x = 1$ and $\int_{-\infty}^{\infty} |g(x)|^2\, \mathrm{d}x = 1$, then, the functions are also *orthonormal*.

Now, let us define the *indicator function* of the set A to be

$$1_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}.$$

Are the following signals orthogonal? Are they orthonormal?

   (a) $e^{i2\pi t} \cdot 1_{[-\pi,\pi]}(t)$ and $1_{[0,1]}(t)$.
   (b) Two arbitrary functions in the form of $\left\{1_{[j,j+1]}(t)\right\}_{j\in\mathbb{Z}}$, where $\mathbb{Z}$ is the set of integers.
   (c) Two arbitrary functions in the form of $\left\{e^{(t-a)/2} \cdot 1_{(-\infty,a]}(t)\right\}_{a\in\mathbb{R}}$.

4. *Fourier Series (6 pts).*
Use the Fourier series backward (synthesis) equation to calculate the time-domain representation of the following signals:

   (a) *Continuous-time.* The period of $x(t)$ is $T = 0.5$ and $c_k = 1/2^{|k|}$. (Hint: Use the geometric series.) Also, express $x(t)$ as an infinite sum of cosines.
   (b) *Discrete-time.* The period of $x[n]$ is $T = 8$. However, a_k has period four, and the first four coefficients are $a_0 = 0$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 0$. Is $x[n]$ real?

5. *Aliasing (3 pts).*
   Give another expression for the discrete-time signal $x[n] = \sin[n]$ in terms of another real-valued sinusoidal function with frequency less than one. Your steps will probably involve first using Euler's identity, then changing both terms using the aliasing property, and finally putting it back together with Euler's identity. Notice that we left 2π out of the argument of $x[n]$ on purpose. The frequency is not one.