Problem Set #6
Not to be handed in.

1. Use the Fourier series synthesis equation to calculate the time-domain representation of the following signals:

   (a) (Continuous-time) The period of $x(t)$ is $T = 0.5$ and $a_k = (1/2)^{|k|}$. (Hint: Use the geometric series.) Also, express $x(t)$ as an infinite sum of cosines.

   (b) (Discrete-time) The period of $x[n]$ is eight. However, a_k has period four, and the first four coefficients are $a_0 = 0$, $a_1 = 0$, $a_2 = -1$, and $a_3 = 0$. Is $x[n]$ real?

2. *Fourier Transform Properties.* In this exercise, you will derive Fourier properties as we did in class. Since these properties are all easily searchable, you must show work to get credit for this exercise. These properties are important for future use, and they will also help illuminate the similarities and differences of the different versions of the Fourier transform (i.e. CTFT, CTFS, DTFT, DTFS).

   (a) Even and Odd signals

      i. Suppose $x(t)$ is an even signal. Show that the CTFT $X(f)$ is even. (Hint: Use the analysis equation and u-substitution.)

      ii. Suppose $x(t)$ is an odd signal. Show that the CTFT $X(f)$ is odd.

      iii. Suppose $x(t)$ is even and real. Show that the CTFT $X(f)$ is even and real. (Hint: Use $X(f) + X(-f)$ with what you known about the transform of even signals, then write them in integral form and combine.)

      iv. Suppose $x(t)$ is odd and real. Show that the CTFT is odd and *imaginary.*
      By combining the previous properties with linearity we see that an even and imaginary signal $x(t)$ transforms to an even and imaginary $X(f)$, and an odd and imaginary signal $x(t)$ transforms to an odd and real $X(f)$. These properties are all true for the other variants of the Fourier transform as well (i.e. CTFS, DTFT, DTFS).

      v. Use the even/odd decomposition of a signal to show that if $x(t)$ is real then $X(f)$ is conjugate symmetric, which means that $X(f) = X^*(-f)$. (Another straightforward proof just involves taking the conjugate of the analysis equation.) What does this property imply about the magnitude and phase of the Fourier transform of a real signal?

   (b) Derivative
   The derivative property of the CTFT derived in lecture is

$$\frac{d}{dt}x(t) \quad \longrightarrow^{\mathcal{F}} \quad i2\pi f X(f).$$

   This property also hold for the CTFS. As an exercise, derive this property (both period and coefficients). That is, assume that $x(t)$ has period T and corresponding coefficients $\{a_k\}$. What is the Fourier series of $\frac{d}{dt}x(t)$?

(c) Difference function

Differentiation is not an operation that works in discrete-time, but we can define a similar difference operation $x_\Delta[n]$ as

$$x_\Delta[n] = x[n] - x[n-1].$$

What is the DTFT of $x_\Delta[n]$ in terms of $X(f)$? How is this similar to the derivative property of the CTFT?

(d) Duality

By inspection of the CTFT equations (analysis and synthesis), we see that Fourier transform pairs can be reversed. That is, if the CTFT of $x(t)$ is $X(f)$, then the Fourier transform of $X(t)$ is $x(-f)$. Verify this by plugging $x(-f)$ into the inverse CTFT (synthesis equation).

This same duality holds in other forms of the Fourier transform. The other easy case is the DTFS. Here we have that if $x[n]$ has period N, and the DTFS coefficients are $\{a_k\}$ (also with period N), then the dual relationship is that the signal $\tilde{x}[n] = a_n$ has Fourier series coefficients $\{x[-k]/N\}$.

What is the dual relationship for the other two cases—CTFS and DTFT? (Hint: Use the synthesis (forward) equation of one with the analysis (inverse) equation for the other.)

3. *Delta functions.*

(a) Sketch $-\delta(t-2) + 3\delta(3t)$.

(b) Sketch $4\delta[n-4] - 2\delta[n+1]$.

(c) Evaluate $\int_{-2\pi}^{0} \cos(t)[\delta(t-\pi) + \delta(t+\pi)]dt$. (Note the limits of the integral!)

(d) Evaluate $\int_{-\infty}^{\infty} e^t \delta(t-u)dt$.

(e) Consider a discrete-time function with $x[-1] = 2$, $x[2] = 5$, $x[3] = 3$, and $x[n] = 0$ for all other n. Write an expression for $x[n]$ in terms of the Kronecker delta function.