Problem Set #7

1. *(5 pts.)* Prove Parsevals Theorem for the continuous-time and discrete-time Fourier transforms. That is, show that the following two identities are valid

$$\int_{-\infty}^{\infty} |x(t)^2| dt = \int_{-\infty}^{\infty} |X(f)^2| df$$

$$\sum_{k=-\infty}^{\infty} |x[k]|^2 = \int_{-1/2}^{1/2} |X(f)^2| df$$

   Use the following steps. First, define $y(t) = x(t) * x^*(-t)$ (this is the autocorrelation function). Show that $y(0)$ is the energy of $x(t)$. Next, derive $Y(f)$ in terms of $X(f)$. Finally, use the synthesis equation (inverse Fourier transform) to express $y(0)$ in terms of $Y(f)$.

2. *(5 pts)* Calculate the DTFT of

$$\text{rect}\left[\frac{n}{N}\right] = u\left[n + \frac{N}{2}\right] u\left[-n + \frac{N}{2}\right]$$

   where N is even, using the geometric series. A change of variables ($m = k + N/2$) will help. Simplify the answer to be expressed only with trigonometric functions (no complex exponentials) using Eulers identity both in the numerator and denominator. Compare this to the CTFT of the continuous-time rect.

3. *(5 pts).* What is the energy of $\text{sinc}^2(t)$?

4. *(5 pts).* Suppose an LTI system has impulse response $h(t) = \frac{1}{3}\text{rect}(t/4)$. Can the output have more energy than the input? Can the output have less energy than the input?

5. *(5 pts).* Define the following signal :

$$x(t) = \begin{cases} t+1, & -1 \le t \le 1 \\ 2, & 1 \le t \le 3 \\ 5-t, & 3 \le t \le 5 \\ 0, & else \end{cases},$$

   Calculate the following, where $X(f)$ is the Fourier transform of $x(t)$. (Hint: You dont need to calculate $X(f)$ for any of these calculations. Start by plotting $x(t)$.)

   (a) Find the phase of $X(f)$ (as a function of f).
   (b) Find $X(0)$.
   (c) Find $\int_{-\infty}^{\infty} X(f)\, \mathrm{d}f$
   (d) Find $\int_{-\infty}^{\infty} X(f) \,\text{sinc}\, f\, e^{i4\pi f}\, \mathrm{d}f$
   (e) Find $\int_{-\infty}^{\infty} |X(f)|^2\, \mathrm{d}f$
   (f) Sketch the inverse Fourier transform of $\text{Re}(X(f))$.