Problem Set #8

1. *(5 pts)* **Sampling:** Suppose $x(t)$ is real and has a non-zero Fourier transform only on the set of frequencies 5-8Hz (both positive and negative). Draw an illustration of the spectrum of $x(t)$. Also draw an illustration of the spectrum when $x(t)$ is sampled at 16Hz. What is the lowest sampling rate from which the signal can still be recovered?

2. *(6 pts)* **Circular Convolution Property:** Consider the following two finite duration discrete-time signals.

$$x[0] = 1, \quad x[1] = 1, \quad x[2] = 0, \quad x[3] = 0.$$  

(a) Calculate the circular convolution of $x[n]$ and $y[n]$.
(b) Calculate the DFT of $x[n]$ and $y[n]$, multiply, and calculate the inverse DFT.

Note: Make sure to use the DFT rather than the DTFS (the difference is the normalization constant.). If you use the DTFS then you need to include the factor of $N = 4$ is the convolution property.

3. *(4 pts)* **Circular Convolution and Multiplication Property:** Suppose $x[n]$ and $y[n]$ have the same duration $N \geq 3$. Their DFTs are given by $a[k]$ and $b[k]$. We know in advance that

$$\sum_{l=0}^{N-1} a[l]b[k-l \mod N] = 1 + e^{-i\frac{4\pi k}{N}}$$

for $k = 0, 1, \cdots, N - 1$.

(a) If $x[n] = 2^{-n}, n = 0, 1, \cdots, N - 1$, calculate $y[n]$. Is $y[n]$ unique?
(b) If $x[n] = 1 + (-1)^n, n = 0, 1, \cdots, N - 1$, redo part (a).
4. **(6 pts) Band Pass Signal Sampling:** Suppose that the Fourier transform of a real signal \( x(t) \) is nonzero only for \( f_1 \leq |f| \leq f_2 \). A naive application of Nyquist Sampling Theorem will lead to a sampling frequency of at least \( 2f_2 \). However, for such a band pass signal, it is possible to sample at a lower frequency and recover \( x(t) \).

\[ X(f) \]

\[ -f_2 \quad -f_1 \quad f_1 \quad f_2 \]

The signal \( x(t) \) is multiplied by a complex exponential function \( e^{i2\pi f_0 t} \) where \( f_0 \) is chosen to be \( \frac{1}{2}(f_2 + f_1) \). The cutoff frequency of the ideal low pass filter is \( \frac{1}{2}(f_2 - f_1) \).

(a) Sketch the Fourier transform of \( x_s(t) \).

(b) Determine the largest possible sample period \( T \) for recovering \( x(t) \).

(c) Design a system that recovers \( x(t) \) from \( x_s(t) \).