Lecture 12

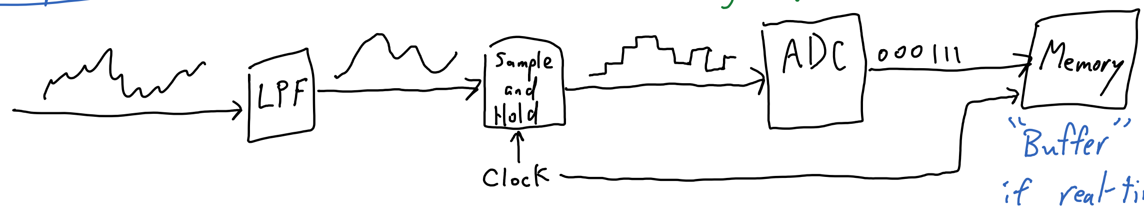
Monday, March 31, 2014
3:39 PM

Analog | **Digital**

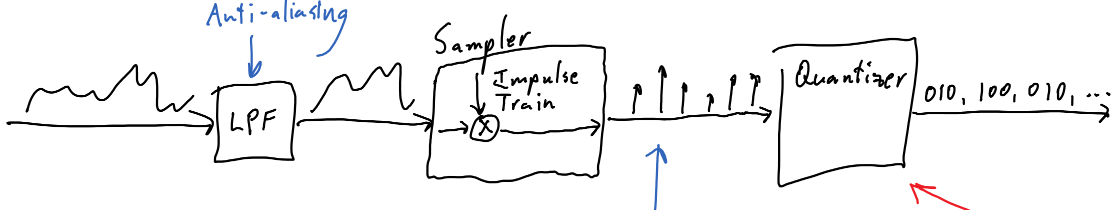
CT
Domain is $\mathbb{R}$ $\xrightarrow{\text{Sampling and Quantization}}$ Finite Space

Implementation:

"Analog to Digital Converter"



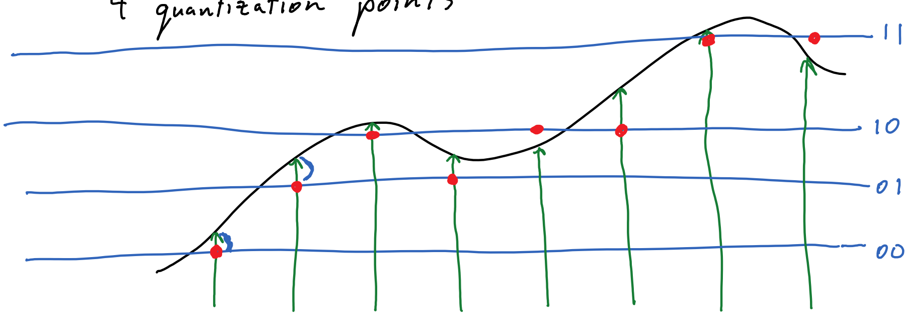
Conceptually:



Lose information.

To this point.
No information is
lost if the signal is bandlimited.

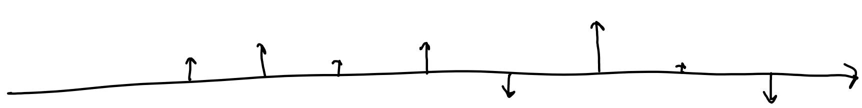
Example: 2-bit quantization
4 quantization points



00, 01, 10, 01, 10, 10, 11, 11

Let $\hat{x}[n]$ be the reconstructed signal (red).
Then $\text{error}[n] = x[n] - \hat{x}[n]$

error[n] "Noise"



CD Audio: 44.1 KHz Sample rate
2 channels (stereo)
16 bit quantization (or 24)

$2^{16} = 65,536$ quant. points

$\Rightarrow 44,100 \times 2 \times 16 = 1,411,200$ bits/sec.
176,400 bytes/sec.
10,584,000 bytes/min.

10 MB/min

Doesn't count overhead.
Not even highest quality.

Probability Primer:

X is a random object.
Represents whether a team wins a game.

$\Omega = \{\text{win}, \text{loss}, \text{tie}\}$

$$\left.\begin{aligned} P(X=\text{win}) &= 70\% \\ P(X=\text{loss}) &= 20\% \\ P(X=\text{tie}) &= 10\% \end{aligned}\right\}$$

Non-negative
Sum to 1.

$P(X=\text{win or } X=\text{loss}) =$
$70\% + 20\% = 90\%$

Random Variable: $\Omega \subseteq \mathbb{R}$

Discrete: X takes one of finitely many (or countably-many) values.

Probability mass function:

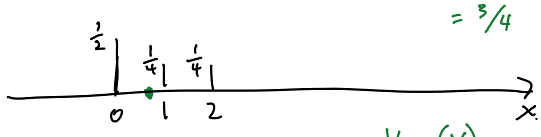
$\text{Prob}(X=a) = p_X(a) \ \forall a.$

$\text{Prob}(X \in S) = \sum_{a \in S} p_X(a)$

Rules:
$p_X(x) \geq 0 \ \forall x$
$\sum_x p_X(x) = 1$

Example:

p_X



$$p_X(x) = \begin{cases} \frac{1}{2}, & x=0 \\ \frac{1}{4}, & x=1,2 \\ 0, & \text{else} \end{cases}$$

$EX = (\frac{1}{2})\cdot 0 + (\frac{1}{4})\cdot 1 + (\frac{1}{4})\cdot 2$
$= 3/4$

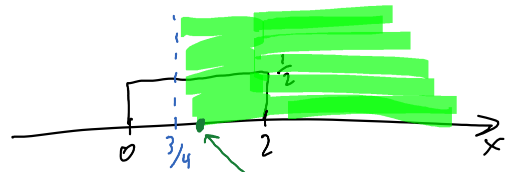
$\text{Var}(X) =$
$(\frac{1}{2})\cdot(\frac{3}{4})^2 + (\frac{1}{4})(\frac{1}{4})^2$
$+ (\frac{1}{4})(\frac{5}{4})^2$
$= \frac{18}{64} + \frac{1}{64} + \frac{25}{64} = \frac{44}{64}$

Continuous: Probability Density Function:

$$f_X(a) \qquad \text{For } S \subset \mathbb{R}, \quad P(X \in S) = \int_S f_X(a)\, da$$

Example: Uniform Distribution:

$$\text{Unif}[0,2] = f_X(a) = \begin{cases} \frac{1}{2}, & 0 \le a \le 2 \\ 0, & \text{else} \end{cases}$$



Rules:

$f_X(x) \ge 0 \quad \forall x$

$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

Prob $\left(X > \frac{3}{4}\right)$

$$EX = \int_{-\infty}^{\infty} f_X(x) \cdot x\, dx$$

$$= \frac{1}{2}\int_0^2 x\, dx = \frac{1}{4}x^2\Big|_0^2 = \frac{4}{4} = 1$$

Expected Value:

$$\text{Var}(X) = \int_{-\infty}^{\infty} f_X(x)(x-1)^2\, dx$$

$$= \frac{1}{2}\int_0^2 (x-1)^2\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} x^2\, dx = \frac{2}{6}$$

Mean of $g(X)$:

$$Eg(X) = \sum_x p_X(x)\, g(x) \leftarrow \text{Discrete}$$

$$Eg(X) = \int_{-\infty}^{\infty} f_X(x)\, g(x)\, dx \leftarrow \text{Continuous}$$

Unif $[a,b]$

$$\text{Var} = \frac{(b-a)^2}{12}$$

Mean of X. First moment.

$$EX = \sum_x p_X(x) \cdot x$$

Law of Large Numbers:

Let $X_1, X_2, \ldots, X_n$ be independent R.V.'s each have dist. p_X

Sample Ave.

$$\frac{1}{n}\sum_{i=1}^{n} g(X_i) \longrightarrow Eg(X)$$

Second Moment: EX^2

$$\text{LLN}: \frac{1}{n}\sum_{i=1}^{n} X_i^2 \rightarrow EX^2 \qquad \text{(Power)}$$

Variance: $E(X - EX)^2$

Standard Deviation: $\sqrt{\text{Var}(X)}$

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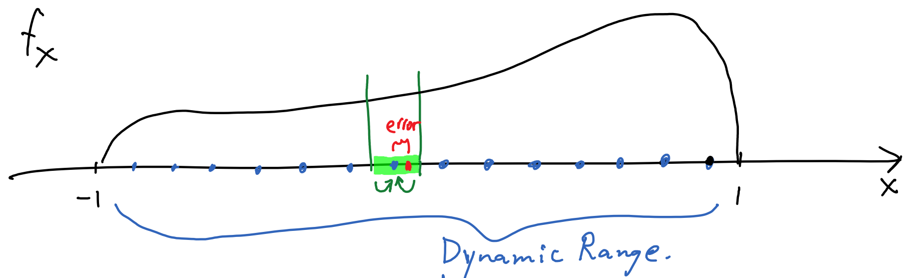
Standard Deviation: $\sqrt{Var(X)}$

Quantization:

Assume a distribution of signal $x[n]$.



Uniform Quantization: Equal spacing:

Error is uniformly distributed if quantization is fine enough.

$$|error| \leq \frac{DR}{2\,(\#\text{ Quant. Points})}$$

$$= \frac{DR}{2\;2^b} \quad \text{where } b \text{ is number of bits for quant.}$$

Power of Quant. Noise (error): $\dfrac{\left(\frac{DR}{2^b}\right)^2}{12}$