Lecture 19
Thursday, April 24, 2014
10:26 AM

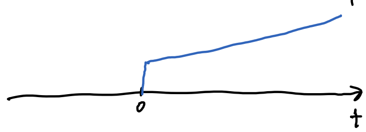
Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

May only converge for some s.

Two parts: 1.) Region of Convergence (ROC)
2.) $X(s)$

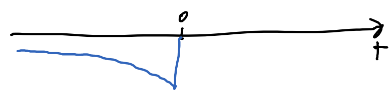
Recall example: $e^{t}u(t) \xrightarrow{\mathcal{L}} \frac{1}{s-1}$



$\text{ROC} = \{s : \text{Re}\{s\} > 1\}$

Notice: No Fourier transform.
Energy is infinite.

$-e^{t}u(-t) \xrightarrow{\mathcal{L}} \frac{1}{s-1}$



ROC: $\text{Re}\{s\} < 1$

Notice: $\text{Re}\{s\} = 0$ is in ROC.
$\Rightarrow$ F.T. exists.

$$\mathcal{F}[x(t)] = X(s)\big|_{s=i2\pi f}$$
$$= \frac{1}{i2\pi f - 1}$$

Inverse of Laplace Transform:

1.) If imag. axis is in the ROC, take the inverse F.T. of $X_L(i2\pi f)$.

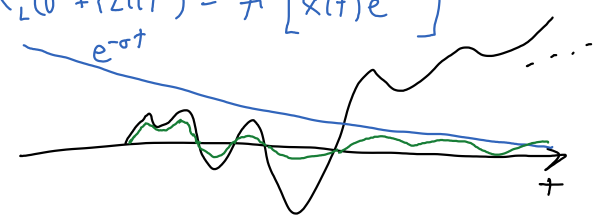
2.) In general:

$$x(t) = \lim_{f\to\infty} \frac{1}{i2\pi} \int_{\sigma - i2\pi f}^{\sigma + i2\pi f} X(s) e^{st}\, ds$$

for any σ that puts this line in the ROC.

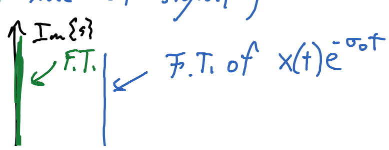
Interpret Laplace Transform:

$$X_L(\sigma + i2\pi f) = \mathcal{F}\left[x(t) e^{-\sigma t}\right]$$

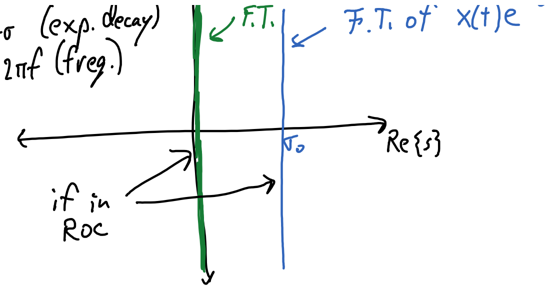


- If σ is large enough, F.T. might converge (finite energy).
- If σ is too large or too small, can cause divergence. (Consider left side of signal)

$\Rightarrow$ $\text{Re}\{s\} = \sigma$ (exp. decay)
$\text{Im}\{s\} = 2\pi f$ (freq.)



$\Rightarrow$ $Re\{s\} = \sigma$ (exp. decay)
$Im\{s\} = 2\pi f$ (freq.)



Laplace trans. is redundant.

Example: $x(t) = rect(t)$

$$X(s) = \int_{-\infty}^{\infty} rect(t) e^{-st} dt$$

$$= \int_{-1/2}^{1/2} e^{-st} dt$$

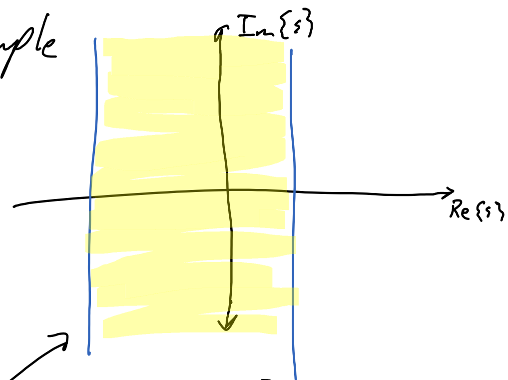
$$= \frac{e^{-s/2} - e^{s/2}}{s}$$

ROC: All s. ROC = complex plane.

Finite-duration:

ROC: Always vertical strips

Example



Left and right boundaries are optional.

Stability: Let $h(t)$ be the impulse response of a system. The system is stable if ROC of $H(s)$ contains the imag. axis in interior. Not on boundary.

Example of unusual case:

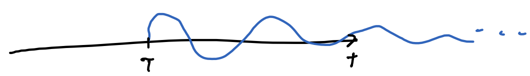
$h(t) = \text{sinc}(t)$

Laplace transform: ROC: $\text{Re}\{s\} = 0$ ← A single vertical strip.

<u>We know this is not stable.</u>

Characterization of signals:

1.) Right-sided signal ⟹ ROC has no right boundary.



$\exists T$ s.t. $x(t) = 0 \;\; \forall t < T$.

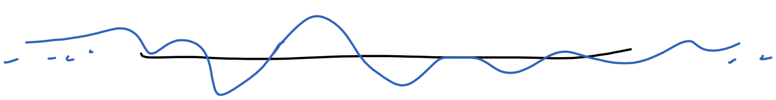
If $\text{Re}\{s\} = \sigma$ is in ROC then $\sigma' > \sigma$ is also in ROC.

2.) Left-sided signal ⟹ ROC has no left boundary.

$\exists T$ s.t. $x(t) = 0 \;\; \forall t > T$.

3.) Two-sided

(unfortunate naming above — neither)



4.) Finite-duration ⟹ ROC has no left or right boundary.

(Mathematically: both left-sided and right-sided.)

(ROC = complex plane)

<u>Causal:</u> ⟹ right-side impulse response.

$h(t) = 0, \;\; \forall t < 0$.

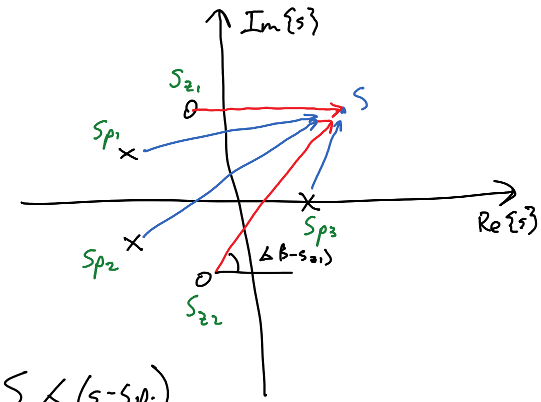
Rational Laplace Transform:

Rational function: $\frac{N(s)}{D(s)}$ ← Polynomial

Factored: $C\frac{(s-s_{z_1})(s-s_{z_2})\cdots}{(s-s_{p_1})(s-s_{p_2})\cdots}$ ← # of term is order of polynomial.

s_{z_i} are the zeros.
s_{p_i} are the poles.

Magnitude: $|C|\frac{|s-s_{z_1}||s-s_{z_2}|\cdots}{|s-s_{p_1}||s-s_{p_2}|}$

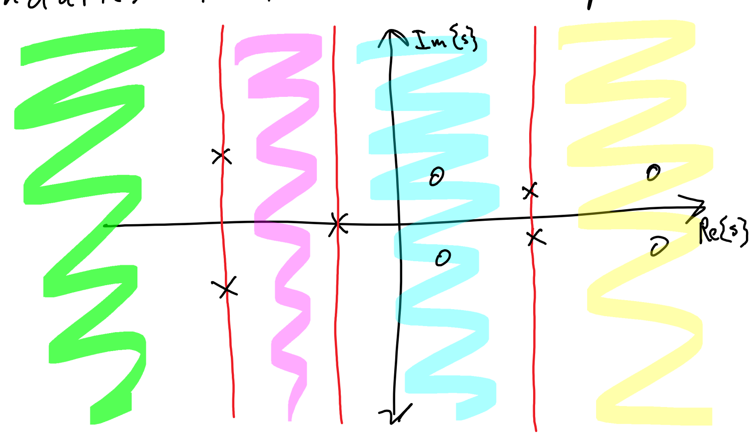


zeros: $|s-s_{z_i}|$
↑ length of vector

Phase: $\measuredangle C + \sum_{\text{zeros}} \measuredangle(s-s_{z_i}) - \sum_{\text{poles}} \measuredangle(s-s_{p_i})$

For Rational Laplace transforms:

Boundaries of ROC are the poles.



Quickly check whether a rational Laplace transform could be both causal and stable.

All poles are in the left half-plane.
(real parts negative)

Z-transform:

Discrete-time signal:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

z is complex.

May converge for some z.

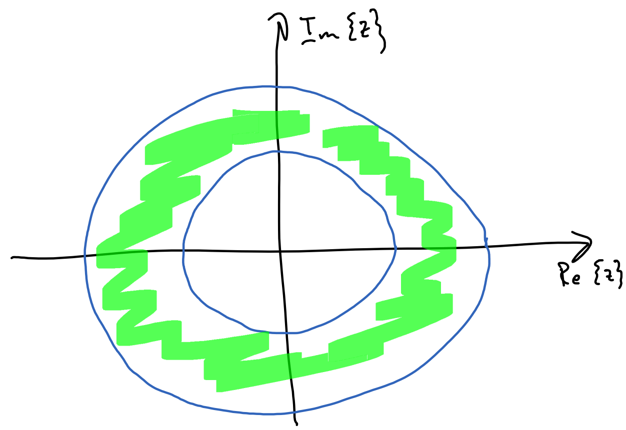
Let $z = a e^{i2\pi f}$

$$X_z\left(a e^{i2\pi f}\right) = \sum_{n=-\infty}^{\infty} x[n]\, a^{-n} e^{-i2\pi f n} = \mathcal{F}\left[x[n]\, a^{-n}\right]$$

$|z| = a$

Relationship to Laplace transform: $z = e^{s}$

ROC are rings:



The s-plane has been wrapped around