Lecture 20

Thursday, April 24, 2014
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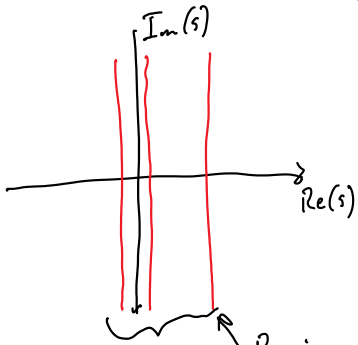
Laplace: $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \mathcal{F}\left(x(t)\, e^{-\sigma t}\right)$, where $\sigma = Re(s)$

$s = \sigma + i2\pi f$

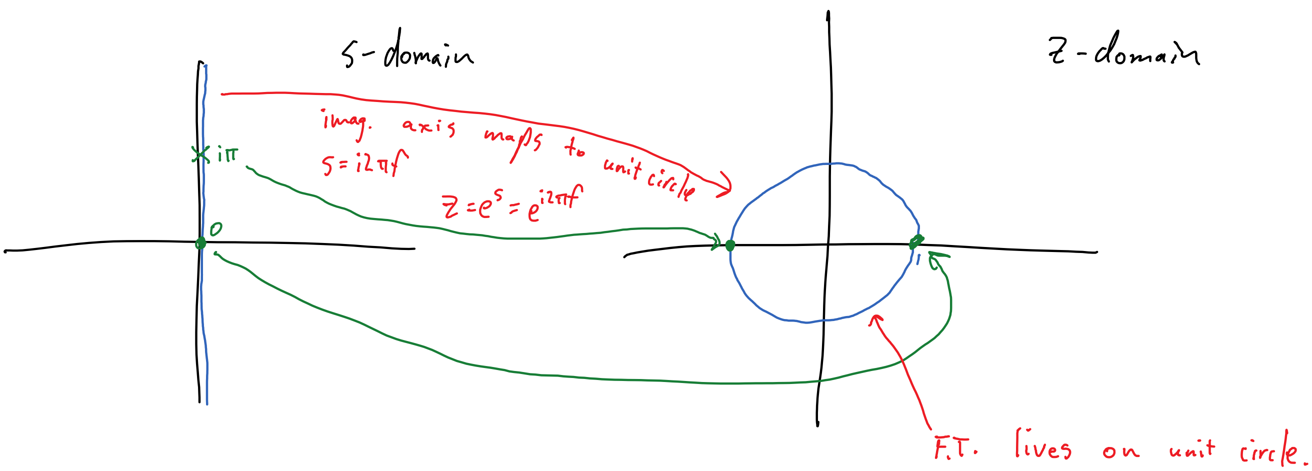


Region of convergence

Z-transform: $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \tilde{\mathcal{F}}\left(x[n]\, a^{n}\right)$

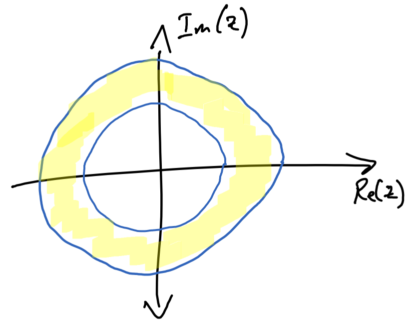
$z = a e^{i2\pi f}$

Substitution: $z = e^{s}$



Region of Convergence:

Rings defined by $|z| \in (a, b)$ Eg.



Right-sided: ROC has no outward boundary.

Right-sided: ROC has no outward boundary.

Left-sided: ROC has no inner boundary (may have pole at $z=0$).

Finite-duration: ROC is the whole space (may be missing $z=0$)

Stability: $\Longrightarrow$ Unit circle is in the ROC.

Stable $\Longleftarrow$ Unit circle is in interior of the ROC

Properties of Laplace and Z-transform:

Linearity : $ax(t) + by(t) \xrightarrow{\mathcal{L}} aX(s) + bY(s)$

Convolution : $x(t) * y(t) \xrightarrow{\mathcal{L}} X(s)\,Y(s)$

Aside: $e^{i2\pi ft} \longrightarrow \boxed{*h}\ (\text{LTI}) \longrightarrow H(f)e^{i2\pi ft}$

$e^{\sigma t}e^{i2\pi ft}$ $\qquad$ F.T. $\qquad H(s)e^{\sigma t}e^{i2\pi ft}$

$s=\sigma+i2\pi f$

Derivative Property: $\frac{d}{dt}x(t) \xrightarrow{\mathcal{L}} s\,X(s)$

Integral Property: $\int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{L}} \frac{1}{s}X(s)$, $\qquad$ ROC : $Re(s) > 0$ intersect with ROC of $X(s)$

Time-shift property: $x(t-T) \xrightarrow{\mathcal{L}} \underline{e^{-Ts}}\,X(s)$

not rational.

D.T. time-shift : $x[n-K] \xrightarrow{\mathcal{Z}} z^{-K}X(z)$

is rational.

Example: $x[n] = \delta[n] - 2\delta[n-1]$

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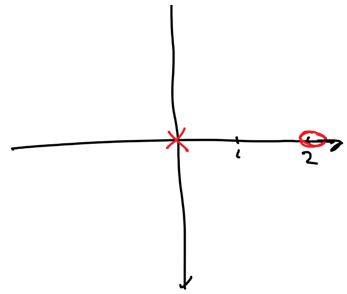
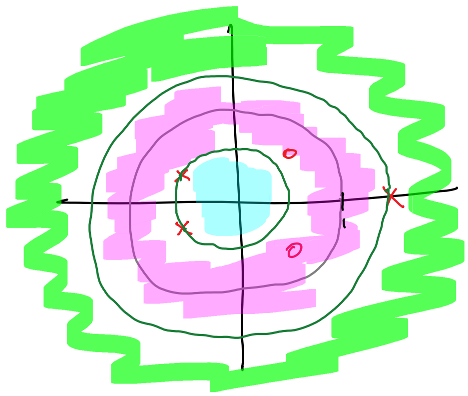
$$X(z) = 1 - 2z^{-1}$$
$$= \frac{z-2}{z}$$

Is this rational?
Yes.



For rational z-transform: - Poles are at the boundaries of ROC
(Boundaries are circles, ROC is connected)



right-sided
not stable.

Neither right or left-sided.
"two sided"
Is stable.

left-sided
Not stable.

Important Transforms to Know:

	Laplace:	
$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re(s) > Re(-a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re(s) < Re(-a)$
$t^K e^{-at}u(t)$	$\frac{K!}{(s+a)^{K+1}}$	$Re(s) > Re(-a)$

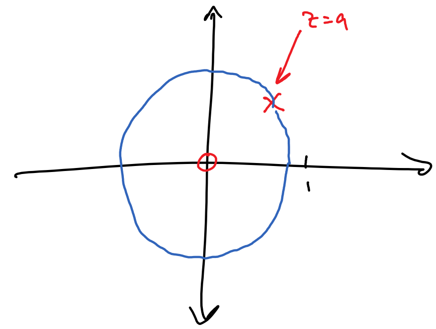
rational

K^{th} order pole (two poles in one place)

	Z-transform:	
$\delta[n]$	1	
$\delta[n-K]$	z^{-K}	
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	same	$\|z\| < \|a\|$

Plot poles and zeros for $x[n] = a^n u[n]$:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$



What about causality?

For rational z-transforms

$\boxed{\text{Right-sided} \Rightarrow \text{causal} \quad \text{iff} \quad \#\text{zeros is} \leq \#\text{poles.}}$

Example of right-sided non-causal:

$x[n] = \delta[n+1] \qquad X(z) = z$

$\#\text{zeros} = 1$
$\#\text{poles} = 0$

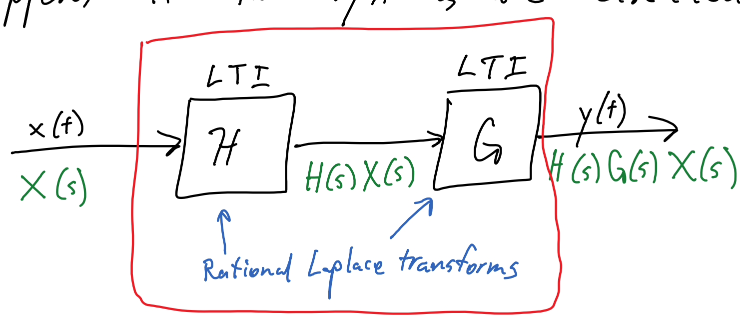
$\sum_{n=-\infty}^{\infty} x[n] z^{-n}$ if $x[K] \neq 0$ for some $K < 0$

$x[K] z^{-K} + \sum_{n \neq K} x[n] z^{-n}$

This is z to a positive power.

order of numerator will always be larger.

What happens if two systems are cascaded:



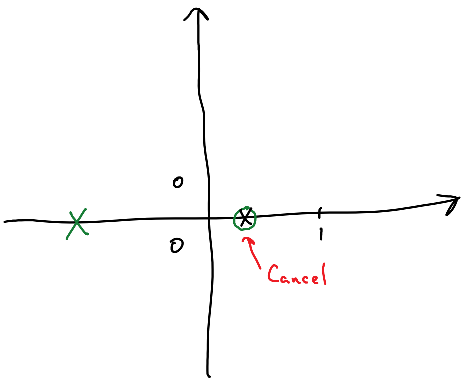
$Y(s) = X(s) \cdot$ (Laplace transform of system)

Transfer Function $\longrightarrow \dfrac{Y(s)}{X(s)} = \underbrace{H(s)\,G(s)}$

Combine poles and zeros

A pole can cancel a zero.

Region of convergence: Intersection of the two ROC's.
If a pole gets canceled, the ROC expands.

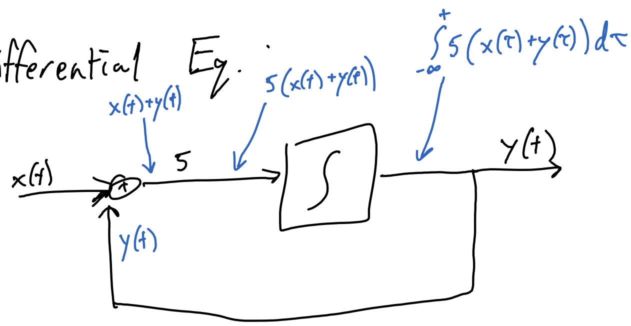


$H(s) \leftarrow$ stable

$G(s) \leftarrow$ unstable

- Only one pole and two zeros remain.
- Result is stable.

Differential Eq.:



$$y(t) = \int_{-\infty}^{t} 5\big(x(\tau)+y(\tau)\big)\,d\tau$$

$$y'(t) = 5\big(x(t)+y(t)\big)$$

$$y(t) = -x(t) + \frac{1}{5}y'(t)$$

Constant Coefficient Linear Differential Eq.:

$$\sum_{k=0}^{M} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Laplace Transform: $$\sum_{k=0}^{M} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{M} a_k s^k}$$

Rational