Lecture 21

Tuesday, April 29, 2014
3:18 PM

Differential Eq.

$$\sum_{k=0}^{M} b^k \frac{d^k}{dt^k} x(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} y(t)$$

$$\sum_{k=0}^{M} b_k s^k X(s) = \sum_{k=0}^{M} a_k s^k Y(s)$$

$$\frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{M} a_k s^k}$$

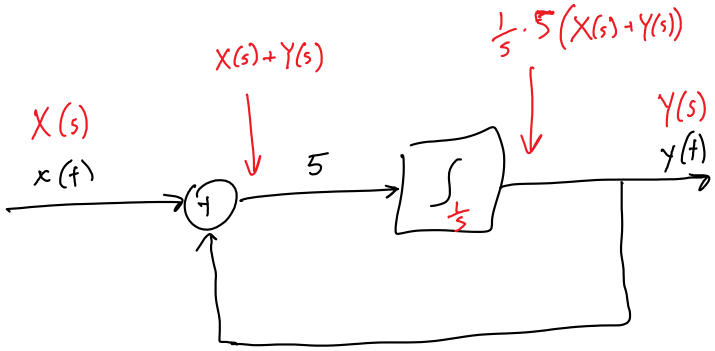
Difference Eq.

$$y[n] = 3x[n] + 2x[n-1] - \frac{1}{2} y[n-1]$$

$$\sum_{k=0}^{M} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{M} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}}$$



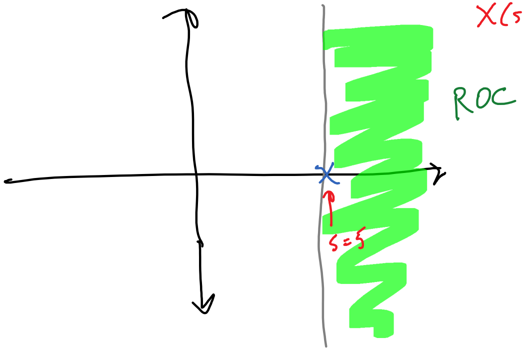
$$Y(s) = \frac{5}{s}(X(s)+Y(s))$$

$$(s-5)Y(s) = 5X(s)$$

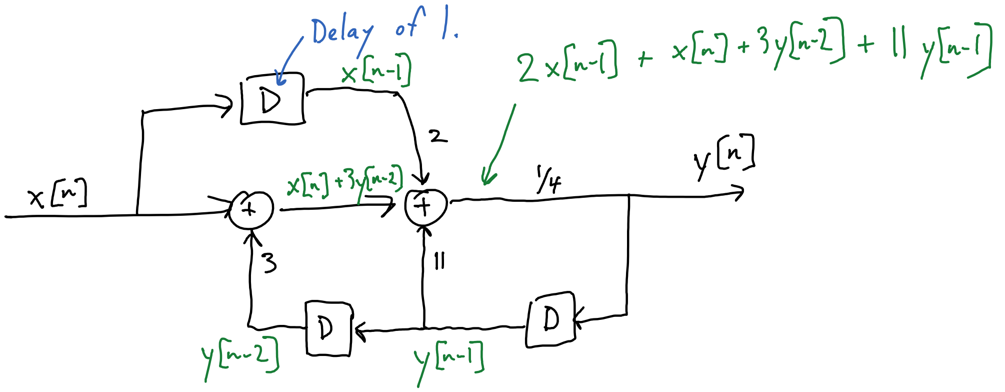
$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{5}{s-5}$$

$$\boxed{5e^{5t}u(t)}$$

Assume causal:



D-T example:

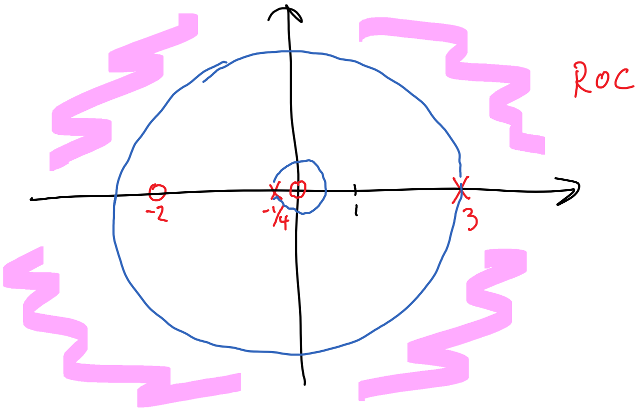


$$y[n] = \frac{1}{4}\left(2x[n-1] + x[n] + 3y[n-2] + 11y[n-1]\right)$$

$$4y[n] - 11y[n-1] - 3y[n-2] = x[n] + 2x[n-1]$$

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{4-11z^{-1}-3z^{-2}} = \frac{z^2+2z}{4z^2-11z-3} = \frac{(z+2)z}{4(z-3)(z+\frac{1}{4})}$$

Assume causal:



Inverse Transform:

Partial Fraction Expansion.

Right-sided z-inverse

$= 1 + \frac{2}{3}$

$$H(z) = \frac{(z+2)z}{4(z-3)(z+\frac{1}{4})} = \frac{(1+2z^{-1})}{\underset{\sim}{4}(1-3z^{-1})(1+\frac{1}{4}z^{-1})} = \frac{A}{1-3z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}}$$

$1 + \frac{1}{12}$

Make the numerator lower order than denom.

Higher-order poles

$$\frac{N(z)}{(1-a_1z^{-1})(1-a_2z^{-1})^2} = \frac{A}{1-a_2z^{-1}} + \frac{B}{(1-a_2z^{-1})^2} + \frac{C}{1-a_1z^{-1}}$$

Multiply out.

$$A = \frac{1+\frac{2}{3}}{4(1+\frac{1}{12})} \qquad B = \frac{1-8}{4(1+12)}$$

$$h[n] = A(3)^n u[n] + B\left(\frac{-1}{4}\right)^n u[n]$$

Long-division: Right-sided inverse. $\longrightarrow$ $\frac{1}{4}\delta[n] + \frac{19}{16}\delta[n-1] + \ldots$

Z-inverse $\uparrow$

$$H(z) = \frac{z^2+2z}{4z^2-11z-3}$$

$$4z^2-11z-3 \overline{\big)\, z^2+2z} \quad \text{quotient: } \frac{1}{4} + \frac{19}{16}z^{-1} + \ldots$$

$$- \quad z^2 - \frac{11}{4}z - \frac{3}{4}$$

$$\frac{19}{\cancel{4}}z + \frac{3}{4}$$

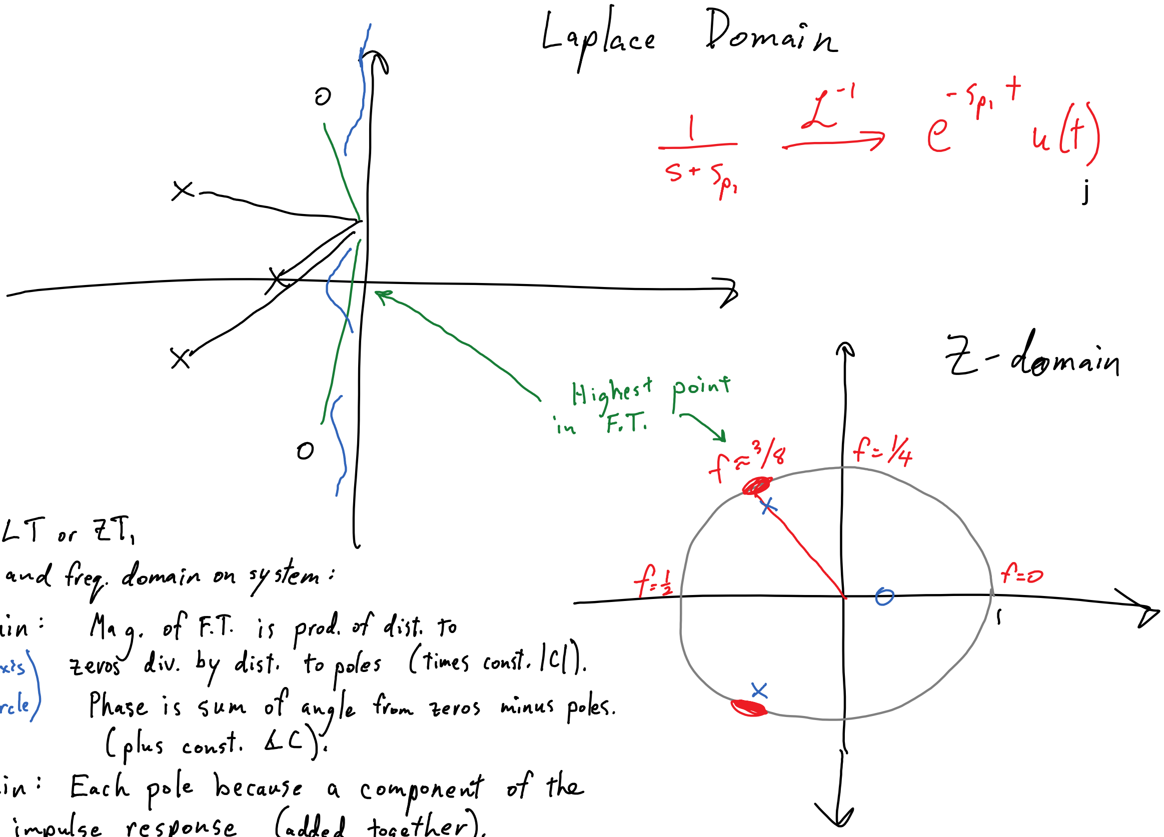
$$- \quad \frac{19}{\cancel{4}}z - \frac{11 \cdot 19}{16} - \frac{3 \cdot 19}{16}z^{-1}$$

$$= \frac{1}{4} + \frac{\frac{19}{4}z + \frac{3}{4}}{4z^2 - 11z - 3}$$

Laplace Domain

$$\frac{1}{s + s_{p_1}} \xrightarrow{\mathcal{L}^{-1}} e^{-s_{p_1} t} u(t)$$

Z-domain

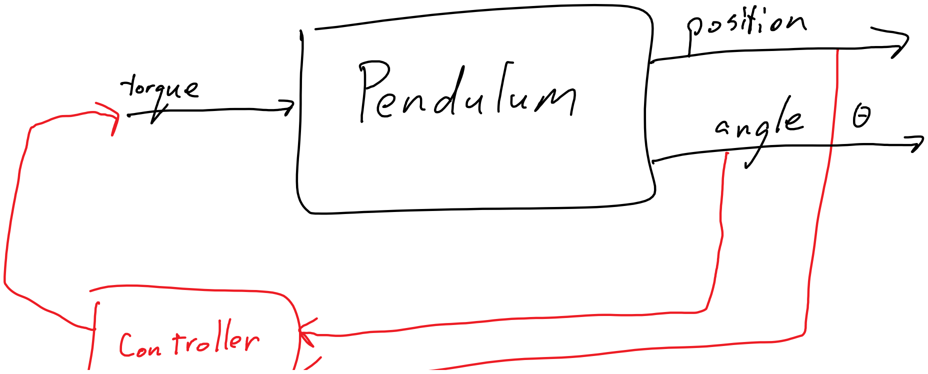


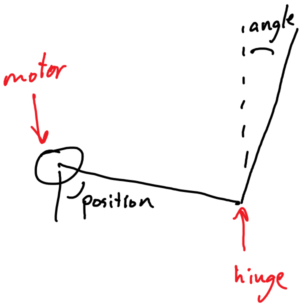
From rational LT or ZT,
interpret time and freq. domain on system:

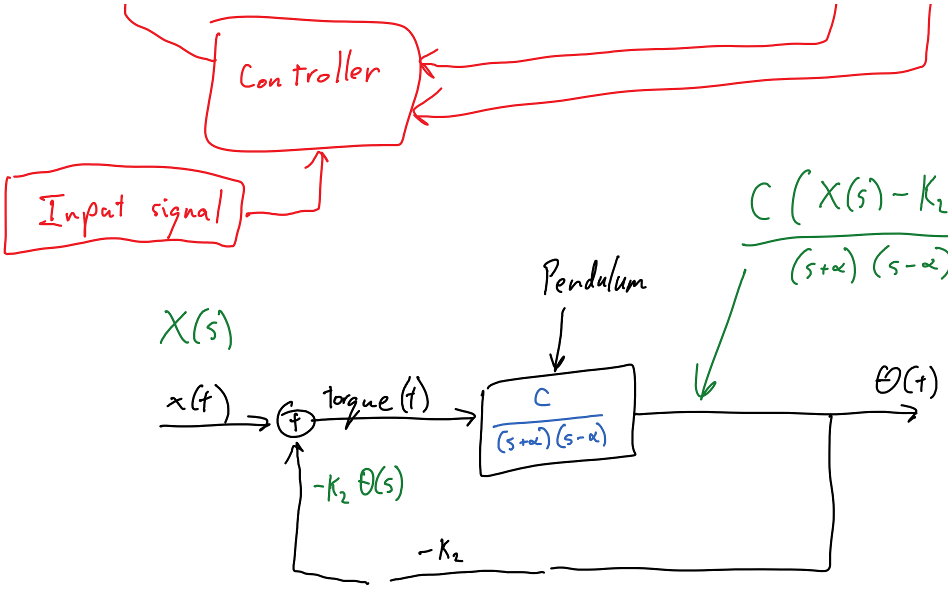
1.) Freq. domain: Mag. of F.T. is prod. of dist. to zeros div. by dist. to poles (times const. $|C|$). Phase is sum of angle from zeros minus poles. (plus const. $\angle C$).

(LT: Imag axis
ZT: Unit Circle)

2.) Time domain: Each pole because a component of the impulse response (added together). The component either grows or decays according to $Re\{s\}$ and oscillates according to $Im\{s\}$, or $|z|$ and $\angle z$ respectively. The ROC determines right-sided or left.







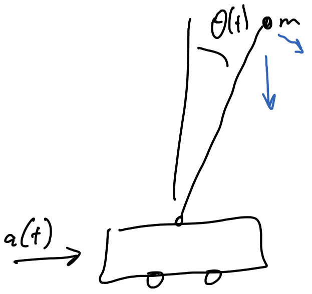
$$\frac{C\left(X(s)-K_2\theta(s)\right)}{(s+\alpha)\,(s-\alpha)} = \theta(s)$$

$$\frac{\theta(s)}{X(s)} = \frac{C}{CK_2+(s+\alpha)(s-\alpha)}$$

$$= \frac{C}{s^2+CK_2-\alpha^2}$$

Poles: $\pm\sqrt{\alpha^2-CK_2}$

P. 11.56



$$g\sin(\theta(t)) - a(t)\cos(\theta(t)) = L\frac{d^2}{dt^2}\theta(t)$$

For small theta: $\sin(\theta(t)) \approx \theta(t)$

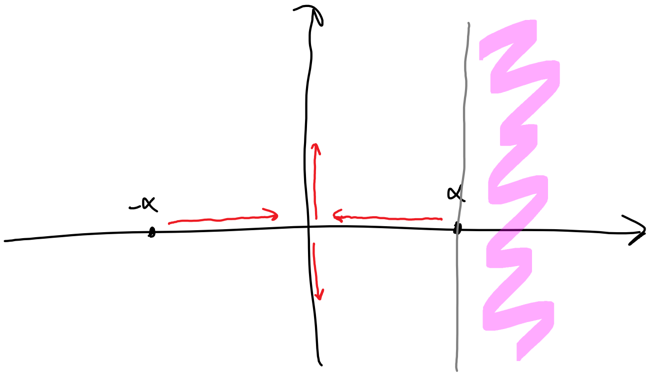
$\cos(\theta(t)) \approx 1$

$$g\,\theta(t) - a(t) = L\frac{d^2}{dt^2}\theta(t)$$

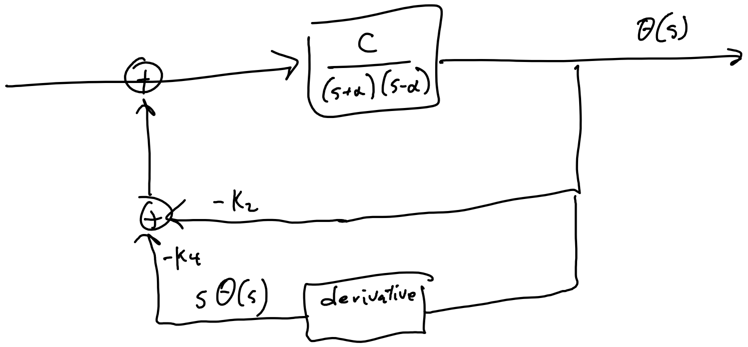
$$g\,\theta(s) - A(s) = L\,s^2\,\theta(s) \Rightarrow (g-Ls^2)\,\theta(s) = A(s)$$

$$\begin{matrix} Y(s) \longrightarrow \\ X(s) \longrightarrow \end{matrix} \frac{\theta(s)}{A(s)} = \frac{1}{g-Ls^2} = \frac{-\frac{1}{L}}{\left(s+\sqrt{\frac{g}{L}}\right)\left(s-\sqrt{\frac{g}{L}}\right)}$$

Poles: $\pm\sqrt{\frac{g}{L}}$

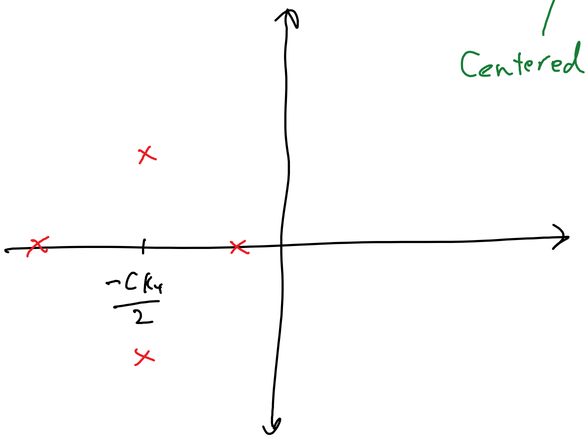


$\alpha = \sqrt{\frac{g}{L}}$



Poles: $$\frac{-CK_4 \pm \sqrt{C^2K_4^2 + 4\alpha^2 - 4CK_2}}{2}$$

Centered around $-\frac{CK_4}{2}$



Use the second sensor in another loop to control position: