Lecture 3

Thursday, February 06, 2014
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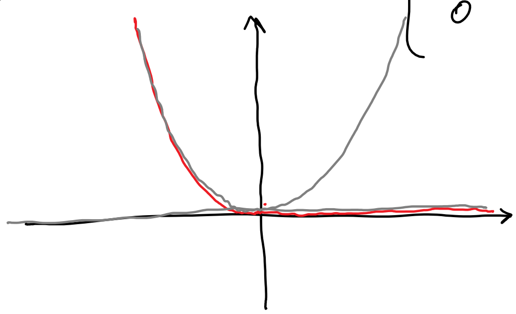
Taylor Series: (t_0)

$$\underline{x(t)} = \underbrace{\overset{\text{Coeffecients}}{x(t_0) + x'(t_0)}(t-t_0) + \frac{1}{2}x''(t_0)(t-t_0)^2 + \dots}_{\text{Polynomial Representation.}}$$

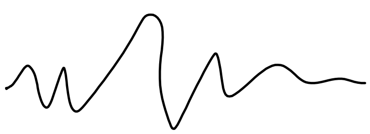
"Analytic" functions are represented this way.
Must be smooth.

Non-analytic signal: $x(t) = \begin{cases} t^2, & t \le 0 \\ 0, & t > 0 \end{cases}$



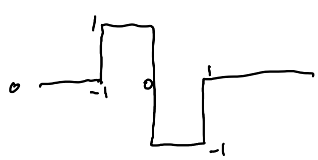
Taylor expantion around $t_0 < 0$: $x_1(t) = t^2$
$t_0 > 0$: $x_2(t) = 0$

<u>Wavelet:</u> Represent signal as linear combination of wavelet.

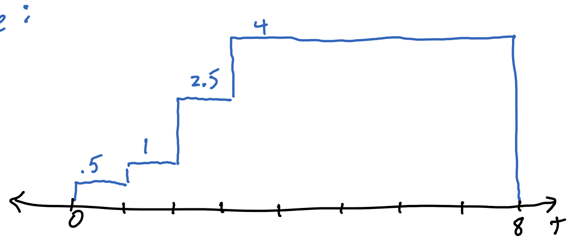


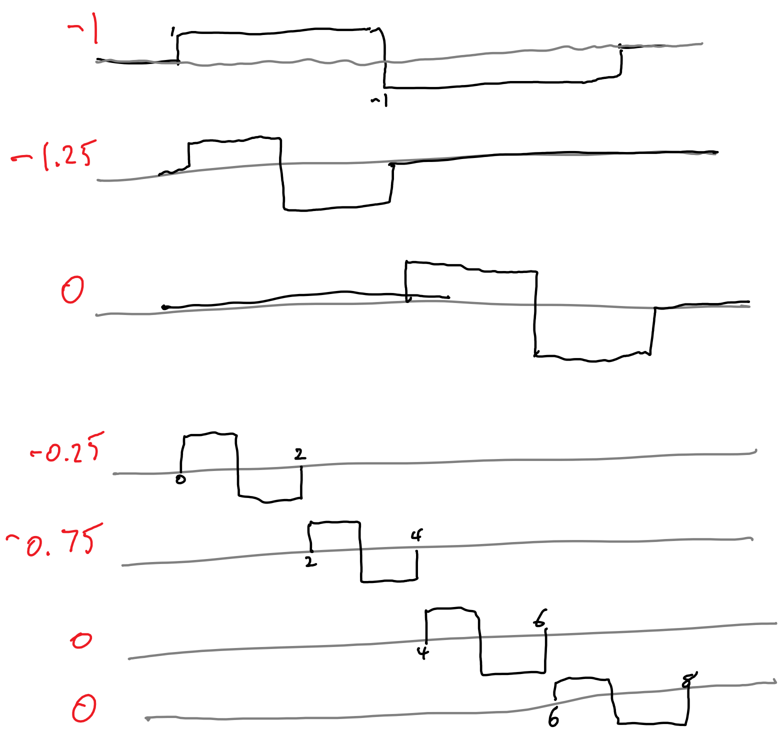
with different shifts and scaling.

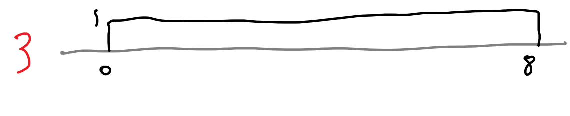
Haar Wavelet:



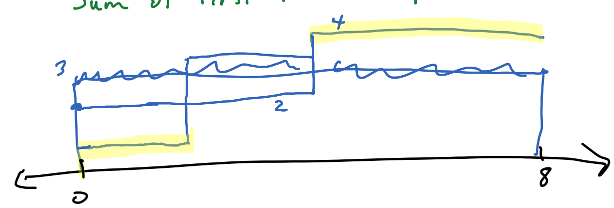
Example:



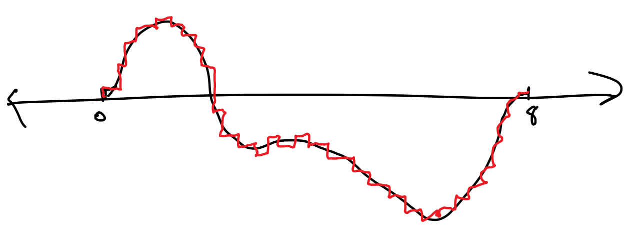




Sum of first three components:



Smooth signals require more refinement to approximate well:



A sparse representation is often very helpful:

(few non-zeros)

Examples:
- Compression
- Pattern Recognition / Machine Learning
- Denoising.

We will see that the Fourier transform is interesting and fundamental for other reasons as well.

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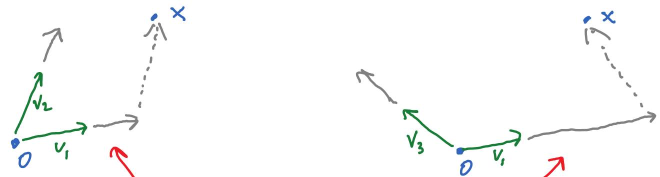
In general, splitting into a linear combination of <u>basis vectors</u> involves interactions between basis vectors.
(Mathematically it's a matrix inverse)

<u>Consider 2-D:</u>



The coefficient for v_1 changed because the other basis vector changed.

<u>Orthonormal basis:</u>

Decomposition is very easy (avoid the above)

$$x[n] = a_1 v_1[n] + a_2 v_2[n] + \ldots = \sum_k a_k v_k[n]$$

If $\{v_k[n]\}$ are orthonormal:

$$a_k = \langle x[n], v_k[n] \rangle$$

$$\Rightarrow x[n] = \sum_k \langle x[n], v_k[n] \rangle \, v_k[n]$$

Let's check:

$$\langle x[n], v_k[n] \rangle = \left\langle \sum_i a_i v_i[n], v_k[n] \right\rangle$$

$$= \sum_i a_i \langle v_i[n], v_k[n] \rangle$$

$$= a_k$$

Fourier Series:

Continous-time: Let $v_k(t) = e^{i2\pi \frac{k}{T} t}$

The $\{v_k(t)\}$ are periodic with period T.

Use the periodic inner product $\langle x, y\rangle_T = \frac{1}{T}\int_0^T x(t)\, y^*(t)\, dt$

Claim: $\{v_k(t)\}$ are orthonormal

$$\langle v_{k_1}(t), v_{k_2}(t)\rangle_T = \frac{1}{T}\int_0^T e^{i2\pi \frac{k_1}{T} t}\left(e^{i2\pi\frac{k_2}{T}t}\right)^* dt$$

$$= \frac{1}{T}\int_0^T e^{i2\pi \frac{k_1-k_2}{T} t}\, dt$$

$$= \begin{cases} \frac{1}{T}\cdot\left[\frac{1}{i2\pi\frac{k_1-k_2}{T}} e^{i2\pi\frac{k_1-k_2}{T}t}\right]_0^T & \text{if } k_1 \neq k_2 \\ 1 & \text{, if } k_1 = k_2 \end{cases}$$

$$= \begin{cases} \underbrace{\frac{1}{i2\pi(k_1-k_2)}\left(e^{i2\pi(k_1-k_2)} - 1\right)}_{=0} & \text{, if } k_1 \neq k_2 \\ 1 & \text{, if } k_1 = k_2 \end{cases}$$

Discrete-time: $v_k[n] = e^{i2\pi\frac{k}{N}n}$ for some integer N.

$\{v_k[n]\}$ are periodic with period N.

Define $\langle x, y\rangle_T = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, y^*[n]$

$\{v_k[n]\}_{k=0}^{N-1}$ are orthonormal:

Check: $\langle v_{k_1}[n], v_{k_2}[n]\rangle = \frac{1}{N}\sum^{N-1} e^{i2\pi\frac{k_1-k_2}{N}n}$

Check: $\langle v_{k_1}[n], v_{k_2}[n] \rangle = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi \frac{k_1-k_2}{N} n}$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(e^{i2\pi \frac{k_1-k_2}{N}} \right)^n$$

constant

Geometric series:

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \Rightarrow \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{iff} \quad |a|<1$$

if $a \neq 1$

if $a=1$: $\sum_{n=0}^{N-1} a^n = N$

$$= \frac{1}{N} \frac{1-\left(e^{i2\pi \frac{k_1-k_2}{\cancel{N}}}\right)^{\cancel{N}}}{1-e^{i2\pi \frac{k_1-k_2}{N}}} \quad \text{if} \quad k_1 \neq k_2$$

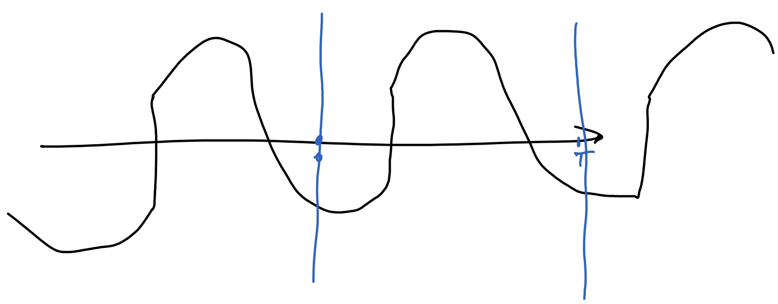
$$= 0$$

Therefore, we arrive at the Fourier Series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{i2\pi \frac{k}{T} t} = \sum_{k=-\infty}^{\infty} a_k v_k(t)$$

$$\Rightarrow a_k = \langle x(t), v_k(t) \rangle_T = \frac{1}{T} \int_0^T x(t)\, v_k^*(t)\, dt$$

$$= \frac{1}{T} \int_0^T x(t) \left(e^{i2\pi \frac{k}{T} t} \right)^* dt$$



Fourier series:

- Periodic
- Finite-duration.