Lecture 4

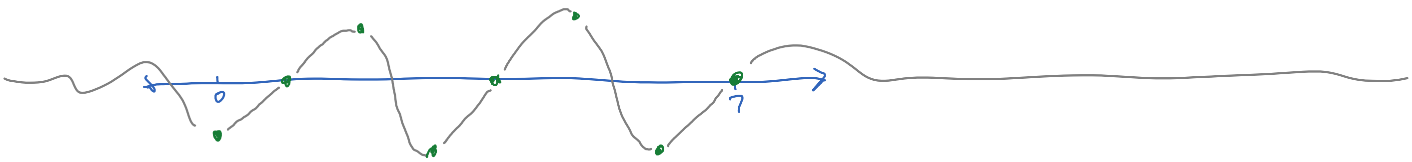
Thursday, February 20, 2014

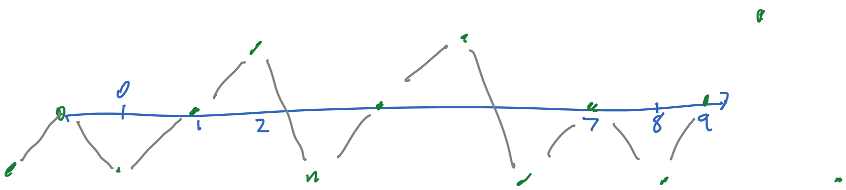
8:39 AM

DFT:

$x = (-1,\ 0,\ 1,\ -1,\ 0,\ 1,\ -1,\ 0)$

$x[0]\quad x[1]\quad \cdots\quad x[8]$





Periodic Extension.

DTFS:

Discrete-time Fourier Series:

$$a_K = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-i\pi \frac{K}{4} n}$$

$N = 8$

$$a_0 = \frac{1}{8}\sum_{n=0}^{7} x[n]$$

$$a_1 = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-i\pi n/4}$$

$$\vdots$$

$$a_7 = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-i\pi n \frac{7}{4}}$$

Frequency Domain: $f = 0$, $f = \frac{1}{8}$, $\vdots$, $f = \frac{7}{8}$

Sampled Signals: 0, $f = \frac{F_s}{8}$, $\vdots$, $f = F_s \cdot \frac{7}{8}$

Try $K = 8$ (not needed):

$$a_8 = \frac{1}{8}\sum_{n=0}^{7} x[n]\, \underbrace{e^{-i\pi n \cdot 2}}_{\left(e^{-i2\pi}\right)^n}$$

$$= \frac{1}{8}\sum_{n=0}^{7} x[n]$$

$$a_9 = \frac{1}{8}\sum_{n=0}^{7} x[n]\, \underbrace{e^{-i\pi \frac{9}{4} n}}_{\cancel{e^{-i2\pi n}}\ e^{-i\pi \frac{1}{4} n}} = a_1$$

$a_{-1} = a_7$

$a_{-2} = a_6$

Hence, **fftshift**

$\cancel{e^{-i2\pi n}} \quad e^{-i\pi\frac{5}{4}n}$

<u>Aliasing</u>: Two different sinusoids can be the same in discrete-time

$$\underbrace{e^{i2\pi f_1 n}}_{\text{signal}} = \underbrace{e^{i2\pi f_2 n}}_{\text{signal}} \quad \text{iff} \quad f_1 - f_2 \text{ is an integer.}$$

$\forall$ integer n

Verify: $\dfrac{e^{i2\pi f_1 n}}{e^{i2\pi f_2 n}} = e^{i2\pi (f_1 - f_2) n} = 1$

<u>DFT</u>: We know CTFS, DTFS } Periodic Signals. Finite-duration signals.

Discrete - Fourier - Transform

Interpret Finite Duration signal as a vector:

$x = (x[0], x[1], \ldots, x[N-1])^T$

$a = (a[0], a[1], \ldots, a[N-1])^T$

$a = DFT(x)$

$x = DFT^{-1}(a)$

Implied period N.

$$a_k = \sum_{m=0}^{N-1} x_m e^{-i2\pi \frac{k}{N} m}$$

$$x_m = \frac{1}{N} \sum_{k=0}^{N-1} a_i e^{i2\pi \frac{m}{N} k}$$

No fundamental difference from DTFS

FFT:

First: Write DFT as matrix multiply:

$$a = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & e^{-i2\pi\frac{1}{N}} & e^{-i2\pi\frac{2}{N}} & \cdots \\ 1 & e^{-i2\pi\frac{2}{N}} & e^{-i2\pi\frac{4}{N}} & \cdots \\ \vdots & & & \end{bmatrix} x$$

$$W_r^k = e^{i2\pi\frac{k}{r}}$$

W_r is an r^{th} root of one.

$$a = \underbrace{\begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & W_N^{-3} \\ W_N^0 & W_N^{-2} & W_N^{-4} & W_N^{-6} & \cdots \\ \vdots & & & \\ W_N^0 & W_N^{-(N-1)} & W_N^{-(N-2)} & \end{bmatrix}}_{\text{DFT Matrix}} x$$

$$W_N^{-(N-1)} = W_N^1 \qquad W_N^{-(N-2)} = W_N^2$$

Cooley-Tukey (1965)

FFT take $n \log n$ time.

FFT = Fast Fourier Transform.

DFT Matrix:

Size $N=2$:

$$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Size $N=4$:

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

This sort of factorization works for all powers of 2.

$F_8 = $ ...

$$F_8 = \underbrace{\begin{bmatrix} \quad & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{8} = \left(\begin{array}{cccc|cccc} 1 & & & & 1 & & & \\ & 1 & & & & W_8^1 & & \\ & & 1 & & & & W_8^2 & \\ & & & 1 & & & & W_8^3 \\ \hline 1 & & & & -1 & & & \\ & 1 & & & & -W_8 & & \\ & & 1 & & & & -W_8^2 & \\ & & & 1 & & & & -W_8^3 \end{array}\right) \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ \hline & & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & 0 & 1 \end{array}\right]$$

Inverse DFT:

$$a = \frac{1}{N} \begin{bmatrix} W_N^0 & W_N^0 & \cdots & - \\ W_N^0 & W_N^1 & W_N^2 & \cdots \\ \vdots & W_N^2 & & \\ \vdots & \vdots & & \end{bmatrix} x$$

$$F_N^{-1} = \frac{1}{N} F_N^*$$