Lecture 5

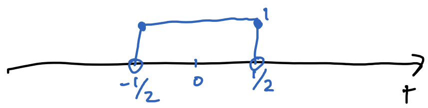
Tuesday, February 25, 2014

1:01 PM

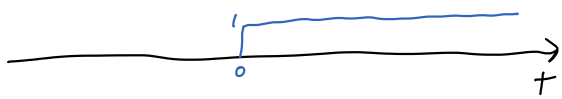
Fourier Transform:

Useful functions:

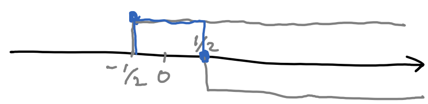
Rect function: $\text{rect}(t) = \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0, & \text{else} \end{cases}$



Unit step function: $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{else} \end{cases}$



Notice: $\text{rect}(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$

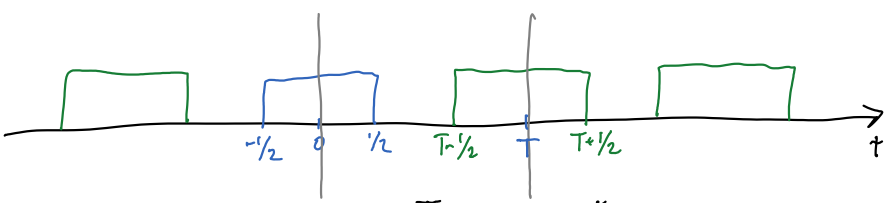


Fourier Series Calculation:

Let $T > 1$

Let $x(t)$ be the periodic expansion of $\text{rect}(t)$

i.e. $x_T(t) = \sum_{m=-\infty}^{\infty} \text{rect}(t - mT)$



CTFS: $a_k = \frac{1}{T} \int_0^T x(t)\, e^{-i2\pi \frac{k}{T} t}\, dt$

Any interval of length T

$$= \frac{1}{T} \int_{-T/2}^{T/2} \underbrace{x(t)}_{\text{rect}}\, e^{-i2\pi \frac{k}{T} t}\, dt$$

$$= \frac{1}{T} \int_{-1/2}^{1/2} e^{-i2\pi \frac{k}{T} t}\, dt \longleftarrow$$ Move unit steps

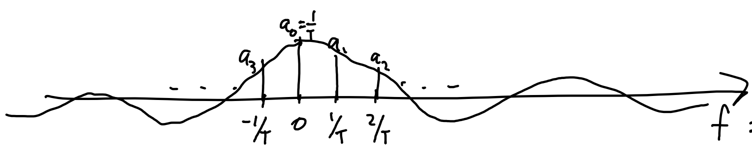
$$= \frac{1}{T}\int_{-1/2}^{1/2} e^{-i2\pi \frac{k}{T} t}\, dt$$

Move unit steps or rect into the limits of integration

$$= \frac{1}{\cancel{T}} \frac{1}{-i2\pi \frac{k}{\cancel{T}}} \left(e^{-i\pi \frac{k}{T}} - e^{i\pi \frac{k}{T}} \right)$$

$$= \frac{1}{i2\pi k} \left(e^{i\pi \frac{k}{T}} - e^{-i\pi \frac{k}{T}} \right)$$

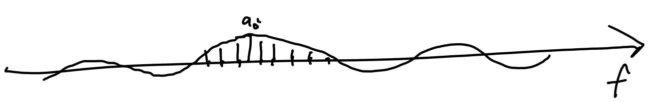
$$= \begin{cases} \dfrac{\sin\left(\pi \frac{k}{T}\right)}{\pi k}, & \text{for } k \neq 0. \\[2ex] \dfrac{1}{T}, & \text{for } k = 0 \end{cases}$$



$f = \frac{k}{T} \Rightarrow k = fT$

Make T larger:

$$a_k = \begin{cases} \dfrac{1}{T}\dfrac{\sin(\pi f)}{\pi f}, & k \neq 0 \\[2ex] \dfrac{1}{T}, & k = 0 \end{cases}$$



Integration: sum and scale by width of interval

Fourier Transform: (continuous-time)

Let $x(t)$ have finite energy:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt$$

Missing $\frac{1}{T}$
Exponent is different
Limits of integration

Inverse: $$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f t}\, df$$
(Synthesis)

First thing to notice: Duality

Forward and Inverse CTFT are (almost) exactly the same.

Let $y(f) = \mathcal{F}\left[x(t)\right]$

then $y(-t) = \mathcal{F}^{-1}\left[x(f)\right]$

Check:

$$z(t) = \int_{-\infty}^{\infty} x(f)\, e^{i2\pi f t}\, df$$

$$z(-\tau) = \int_{-\infty}^{\infty} x(f)\, e^{-i2\pi f \tau}\, df$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi \tau t}\, dt = \mathcal{F}\left[x(t)\right]$$

$x \xrightarrow{\mathcal{F}} X \xrightarrow{\text{Duality}} X \xrightarrow{\mathcal{F}} x$ flipped horizontally

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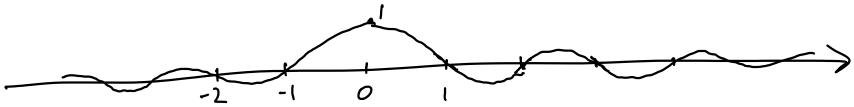
Fourier Transform of $x(t) = \text{rect}(t)$:

$$X(f) = \int_{-\infty}^{\infty} \text{rect}(t)\, e^{-i2\pi f t}\, dt$$

$$= \int_{-1/2}^{1/2} e^{-i2\pi f t}\, dt$$

$$= \frac{1}{-i2\pi f}\left(e^{-i\pi f} - e^{i\pi f}\right)$$

$$= \frac{\sin(\pi f)}{\pi f} = \underline{\mathrm{sinc}(f)} \longleftarrow \text{Definition.}$$

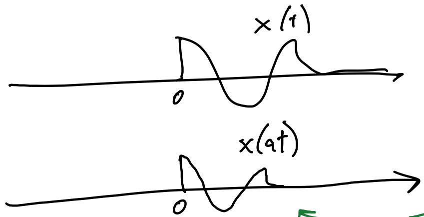


By duality: $\boxed{\mathrm{sinc}(t) \xrightarrow{\mathcal{F}} \mathrm{rect}(f)}$

Time scaling property:

$x(at)$

$\uparrow$ real number



$\longleftarrow$ Example with $a>1$

Assume $X(f)$ is CTFT of $x(t)$

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-i2\pi f t}\, dt$$

Let $\tau = at$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi \left(\frac{f}{a}\right)\tau}\, d\tau$$

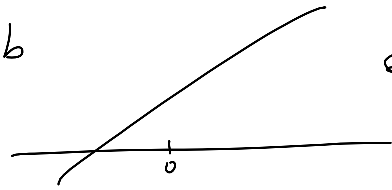
$$= \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Linearity:

Two requirements:

- Superposition Property $\quad G(x+y) = G(x) + G(y)$
- Scaling Property $\quad G(ax) = a\,G(x)$

$y = ax + b$



← Affine but not linear

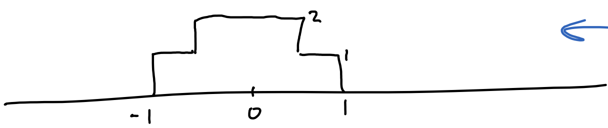
Linearity of Fourier Transform:

$$\mathcal{F}[x(t) + y(t)] = \int_{-\infty}^{\infty} (x(t) + y(t))\, e^{-i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt + \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi f t}\, dt$$

$$= \mathcal{F}[x(t)] + \mathcal{F}[y(t)]$$

Scaling also works.



← Decompose into rects.

Time-shift Property:

$$\mathcal{F}[x(t-\tau)] = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-i2\pi f t}\, dt$$

$u = t - \tau$

$$= \int_{-\infty}^{\infty} x(u)\, e^{-i2\pi f (u+\tau)}\, du$$

$$= e^{-i2\pi f \tau} \int_{-\infty}^{\infty} x(u)\, e^{-i2\pi f u}\, du$$

$$= \underline{e^{-i2\pi f \tau}}\, X(f)$$

Phase

$\angle X(f) - 2\pi f \tau$

Let $\angle X(f)$ be the phase of $X(f)$

line

DTFT

(discrete-time)

$x[n]$

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi fn}$$ Not discrete-frequency

$$x[n] = \int_{-1/2}^{1/2} X(f) e^{i2\pi fn} df$$

$X(f)$ is periodic because of aliasing

period 1.

Text uses angular frequencies:

$\sin(\omega t)$

Fourier Transform: $$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$