Lecture 6

Wednesday, February 26, 2014
10:37 PM

Systems

Systems map input (signal) to output (signal).

noisy image $\xrightarrow{\text{filter}}$ denoised image

trades $\xrightarrow{\text{market}}$ price

sensor $\longrightarrow$ actuator

$$x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$$

Notation: $y(t) = H(x(t))$ ← really a signal

System Properties

– **Memoryless**: H is memoryless if $y(t) = H(x(t))$ depends only on current input $x(t)$.

Eg:
$$y(t) = \sin(t) + \underline{x(t)} \quad ✓$$
$$y(t) = \sin(t-1) + \underline{x(t)} \quad ✓$$
$$y(t) = \int_0^t x(s)\,ds \quad ✗$$
$$y(t) = \int_{-t}^{t} x(s)\,ds \quad ✗$$

– **Causal**: Say H is causal if $y(T)$ depends only on $x(t)$, $\forall\, t \le T$.

If $x_1(t) = x_2(t)$, $\forall\, t \le T$,

Then $\underset{\overset{\shortparallel}{H(x_1(t))}}{y_1(t)} = \underset{\overset{\shortparallel}{H(x_2(t))}}{y_2(t)}$, $\forall\, t \le T$.

Note: Memoryless $\Rightarrow$ Causal.

- **Time-Invariance**: $\mathcal{H}$ is T.I. if

$$y(t) = \mathcal{H}(x(t)) \Rightarrow y(t-t_0) = \mathcal{H}(x(t-t_0)), \ \forall t_0.$$

Eg: $y(t) = \sin(t) + x(t) \xrightarrow{\text{Time Shift}} \sin(t-t_0) + x(t-t_0)$

$x_{NEW}(t) = x(t-t_0)$

$\Rightarrow y_{NEW}(t) = \sin(t) + x_{NEW}(t)$

$= \sin(t) + x(t-t_0)$

$y(t) = x(t) + x(t+1)$

- **Linear**: Say $\mathcal{H}$ is linear if $\left.\begin{array}{l} x_1(t) \xrightarrow{\mathcal{H}} y_1(t) \\ x_2(t) \xrightarrow{\mathcal{H}} y_2(t) \end{array}\right\} \Rightarrow a \cdot x_1(t) + b \cdot x_2(t) \xrightarrow{\mathcal{H}} a \cdot y_1(t) + b \cdot y_2(t).$

for any constants a, b.

Eg: Fourier transform.

- **Invertible**: $\mathcal{H}$ is invertible if you can recover $x(t)$ from $y(t) = \mathcal{H}(x(t))$.

No loss of information. " $\mathcal{H}^{-1}$ exists "

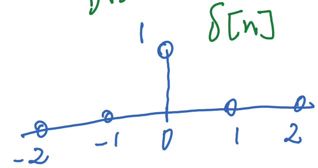
- **Stable**: $\mathcal{H}$ is stable if $x(t) \le B_x, \forall t$, then $y(t) = \mathcal{H}(x(t)) \le B_y, \forall t$.

Output cannot grow too large.

LTI Systems

Impulse Response — Fingerprint of System

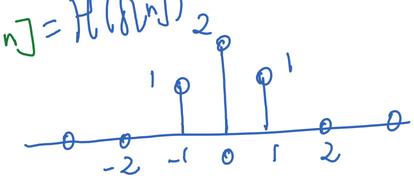
(Kronecker) Discrete delta function $\delta[n]$



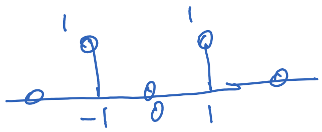
The impulse response of $\mathcal{H}$ is

$$h[n] = \mathcal{H}(\delta[n]).$$

Eg: $h[n] = \mathcal{H}(\delta[n])$ — LTI System



$x[n]$



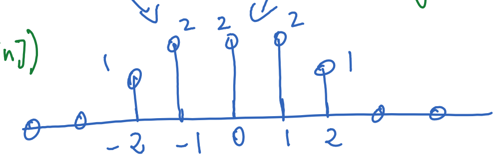
$= \delta[n+1] + \delta[n-1]$

T-I

$y[n] = \mathcal{H}(x[n])$

Linearity

$y[n] = \mathcal{H}(x[n])$

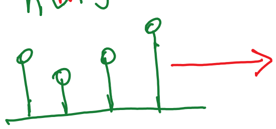


For any LTI system,

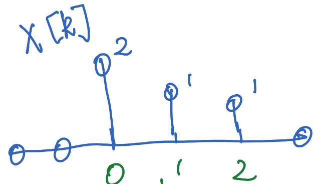
$$\begin{aligned}
y[n] &= \mathcal{H}(x[n]) \\
&= \mathcal{H}\left(\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]\,\mathcal{H}(\delta[n-k]) && \text{Linearity} \\
&= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k] && \text{Time Invariance} \\
&= (x * h)[n]
\end{aligned}$$

Convolution: $(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$.

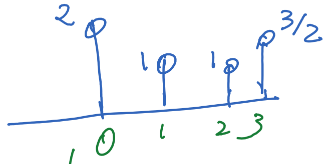
$h[n-k]$



$x[k]$



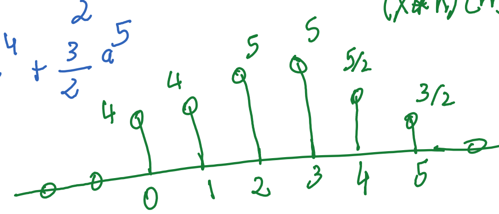
$h[k]$



Polynomial Trick: $(2\cdot a^0 + 1\cdot a^1 + 1\cdot a^2)\left(2\cdot a^0 + 1\cdot a^1 + 1\cdot a^2 + \frac{3}{2}\cdot a^3\right)$

$$\begin{aligned}
&= 4 + 2a + 2a^2 + 3a^3 \\
&\quad + 2a + a^2 + a^3 + \tfrac{3}{2}a^4 \\
&\quad + 2a^2 + a^3 + a^4 + \tfrac{3}{2}a^5 \\
&= 4 + 4a + 5a^2 + 5a^3 + \tfrac{5}{2}a^4 + \tfrac{3}{2}a^5
\end{aligned}$$

$(x*h)[n]$



(DT) Fourier Transform and Convolution

$$\mathcal{F}[y[n]] = \sum_{n=-\infty}^{\infty} y[n]\, e^{-i2\pi f n}$$ (Analysis)

$y[n] = \mathcal{H}(x[n])$

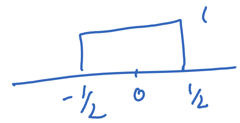
$$= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right) e^{-i2\pi f k}\, e^{-i2\pi f (n-k)}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x[k]\, e^{-i2\pi f k}}_{\mathcal{F}[x[n]]} \; \underbrace{\sum_{n=-\infty}^{\infty} h[n-k]\, e^{-i2\pi f (n-k)}}_{\mathcal{F}[h[n]]}$$

Convolution Property

$$\mathcal{F}[(x * h)[n]] = \mathcal{F}[x[n]] \cdot \mathcal{F}[h[n]].$$

rect(f)



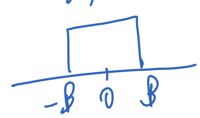
Application: -) Efficient multiplication of polynomials. Reduction from N^2 operations to $N \cdot \log N$ operations.

-) Filtering

X(f)

×

H(f)



Equivalent to convolution with sinc(f).