Lecture 7

Tuesday, March 04, 2014
1:30 PM

Continuous-time Delta Function (Dirac delta function)

Not as straightforward as the discrete-time delta function

(Kronecker delta function)

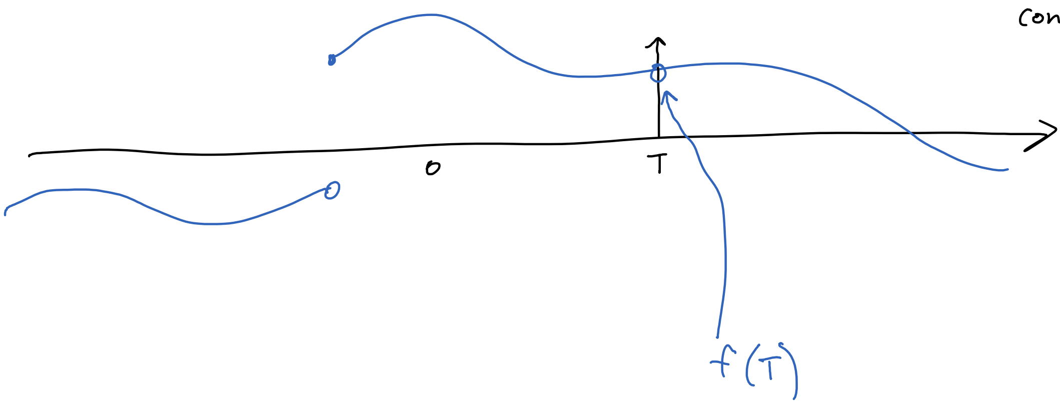
$$1_{\{0\}}[n] = \delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$$

$\delta(t)$ is not actually a function.
We pretend it is and define how it behaves under the integral.

Define: $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, f(t)\, dt = f(0)$ if f is continuous at $t=0$.

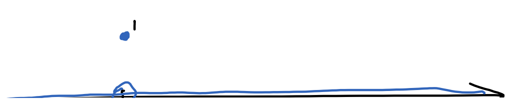
<u>Shifting Property</u>: $\displaystyle\int_{-\infty}^{\infty} \delta(t-T)\, f(t)\, dt$ $\qquad \tau = t-T$

$$= \int_{-\infty}^{\infty} \delta(\tau)\, f(\tau+T)\, d\tau = f(T)$$ if $f(t)$ is continuous at $t=T$

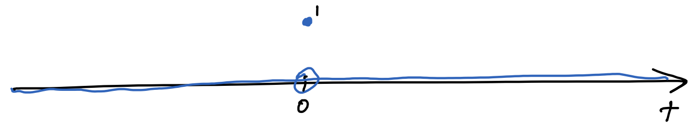


So what is $\displaystyle\int_{-\infty}^{\infty} \delta(t)\, dt$? $= 1$

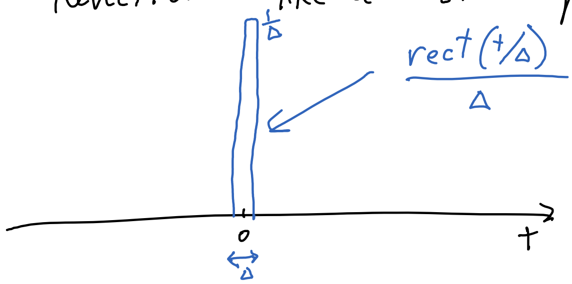
Is it different from $1_{\{0\}}(t)$



Very different from $1_{\{0\}}(t)$



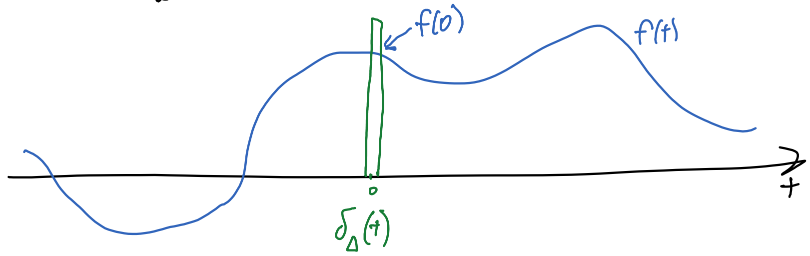
Delta function like a short pulse:



$$\frac{\mathrm{rect}(t/\Delta)}{\Delta} \triangleq \delta_\Delta(t)$$

This function $\delta_\Delta(t)$ behaves like $\delta(t)$ if Δ is small.

Precisely: $\lim\limits_{\Delta \to 0} \int_{-\infty}^{\infty} \delta_\Delta(t)\, f(t)\, dt = f(0)$ if $f(t)$ is continuous at $t=0$.



Tempting to say $\lim\limits_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$

Not true: $\lim\limits_{\Delta \to 0} \delta_\Delta(t) = \begin{cases} 0, & \forall t \neq 0 \\ \infty, & t = 0 \end{cases}$

More importantly, the function does not converge in L_2

Behavior under the integral converges:

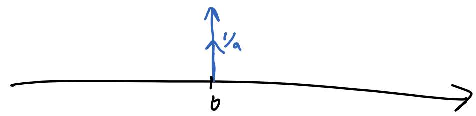
Mathematical precision: $\int d\mu$ ← measure

It is convenient to interchange:

$$\delta(t-T)\, f(t) = \delta(t-T)\, f(T)$$

$$\int^{\infty} \downarrow \, dt = f(T) = \int \downarrow \, dt$$

$$\int_{-\infty}^{\infty} \downarrow \, dt = f(T) = \int \downarrow \, dt$$



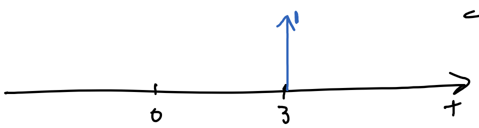
$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Check: $\int_{-\infty}^{\infty} \delta(at)\, dt$ $\tau = at$

$$\frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau)\, d\tau = \frac{1}{|a|}$$

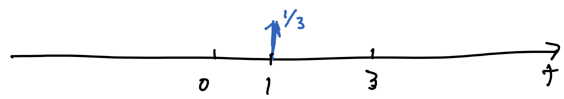
$$\int_{-\infty}^{\infty} \delta(at) f(t)\, dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau) f\left(\frac{\tau}{a}\right) d\tau = \frac{1}{|a|} f(0)$$

$\delta(t-3)$



Scale horizontally $\longrightarrow$ by factor of 3.

$\delta(3t-3)$



Check: $\delta(3t-3) = \delta(3(t-1))$

$$= \frac{1}{3}\delta(t-1)$$

What about integrating over a restricted set.

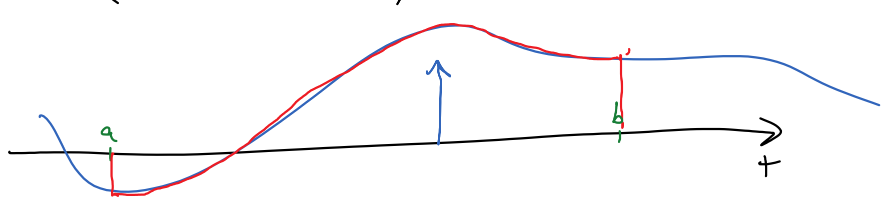
$$\int_A \delta(t) f(t)\, dt = \int_{-\infty}^{\infty} \delta(t)\left(f(t)\, 1_A(t)\right) dt$$

eg. $\int_a^b \delta(t-T) f(t)\, dt = \begin{cases} f(T) & \text{if } T \in (a,b) \\ 0, & \text{else} \end{cases}$

Delta functions in the integration interval are all that matter.

Undefined if right on the edge or f is discontinuous.

$$= \int_{-\infty}^{\infty} \delta(t-T)\Big(f(t)\big(u(t-a) - u(t-b)\big)\Big) dt$$



What is $\int_{-\infty}^{t} \delta(\tau)d\tau$?

$= u(t)$

$u'(t) = \delta(t)$

This gives us a way of talking about derivatives at discontinuities

What is the energy of $\delta(t)$?

$$\int_{-\infty}^{\infty} \delta^2(t)\, dt = \infty$$

Consider rect approximation.

$E\{\delta_\Delta(t)\} = \frac{1}{\Delta}$

We do not have a way to deal with products of delta functions.

But we have a trick for convolution.

Convolution:

Let $x[n]$ and $y[n]$ be DT signals:

$$(x*y)[n] = \sum_{k=-\infty}^{\infty} x[k]\,y[n-k]$$

Let $x(t)$ and $y(t)$ by CT signals:

$$(x*y)(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$$

We care about convolution because of LTI systems.

The behavior of an LTI system H is completely specified by an impulse response $h(t)$:

$$H[x(t)] = (x*h)(t)$$

$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \xrightarrow{(x*h)(t)}$ Equalently $\xrightarrow{x(t)} \boxed{*h} \xrightarrow{(x*h)(t)}$

What happens in this case?

$\xrightarrow{\delta(t)} \boxed{\begin{matrix}\text{LTI}\\ H\end{matrix}} \xrightarrow{?}$

$h(t)$

$$(\delta * h)(t) = \int_{-\infty}^{\infty} \delta(\tau)\, h(t-\tau)\, d\tau$$
$$= h(t)$$

This motivates us to call $h(t)$ the impulse response.

A common abuse of notation:

$(x * y)(t)$ written as $x(t) * y(t)$

- Reason this is convenient: We can concisely write convolutions of manipulated signals

  e.g. $x(t-T) * y(3t)$

  Without this notation:

  $x_2(t) = x(t-T)$

  $y_2(t) = y(3t)$

  $(x_2 * y_2)(t)$

- Reason this is confusing: Convolution is a function of the entire signal, not just at some time t.

Verify CT Convolution:

$$\mathcal{H}(x(t)) = \mathcal{H}\left(\int_{-\infty}^{\infty} \delta(\tau - t)\, x(\tau)\, d\tau\right)$$
$$= \mathcal{H}\left(\int_{-\infty}^{\infty} \delta(t-\tau)\, x(\tau)\, d\tau\right)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, \mathcal{H}\left(\delta(t-\tau)\right) d\tau \quad \text{Linearity}$$
$$= \int x(\tau)\, h(t-\tau)\, d\tau \quad \text{Time invariance.}$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$ Time invariance.

Convolution Commutes:

$$x * y = y * x$$