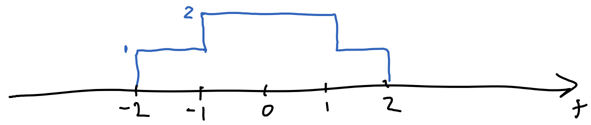
Lecture 9

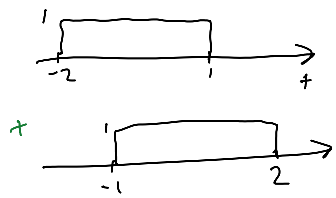
Saturday, March 08, 2014
12:57 PM

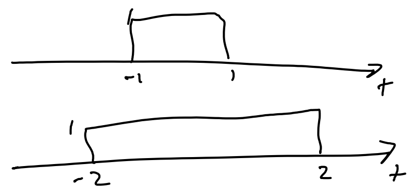
FT Example:

$x(t)$:



$x(t) = \text{rect}(t/2) + \text{rect}(t/4)$





$$x(t) = \text{rect}\left(\frac{t+\frac{1}{2}}{3}\right) + \text{rect}\left(\frac{t-\frac{1}{2}}{3}\right) = \text{rect}\left(\frac{t}{3}+\frac{1}{6}\right) + \text{rect}\left(\frac{t}{3}-\frac{1}{6}\right)$$

$$\text{rect}(t) \to \text{sinc}(f)$$

$$\Rightarrow \text{rect}(t/3) \to 3\,\text{sinc}(3f)$$

$$\Rightarrow \text{rect}\left(\frac{t+\frac{1}{2}}{3}\right) \to 3e^{i2\pi(\frac{1}{2})f}\,\text{sinc}(3f)$$

$$\text{rect}(t) \to \text{sinc}(f)$$

Shift by $\frac{1}{6}$: $\text{rect}(t+\frac{1}{6}) \to e^{i2\pi(\frac{1}{6})f}\,\text{sinc}(f)$

Scale: $\text{rect}(\frac{t}{3}+\frac{1}{6}) \to 3e^{i2\pi(\frac{1}{6})(3f)}\,\text{sinc}(3f)$

$$X(f) = 3\,\text{sinc}(3f)\left(e^{i\pi f} + e^{-i\pi f}\right)$$

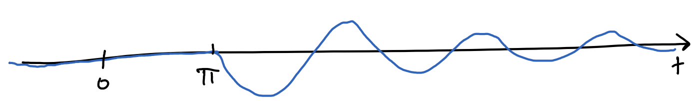
$$= 6\,\text{sinc}(3f)\cos(\pi f)$$

$$x(t) = \text{rect}(t/2) + \text{rect}(t/4)$$

$$\Rightarrow X(f) = 2\,\text{sinc}(2f) + 4\,\text{sinc}(4f)$$

$$x(t) = \underline{u(t-\pi)\,e^{-2t}}\ \underline{\sin(t)}$$

$$\sin(t) \xrightarrow{\mathcal{F}} \frac{1}{2i}\delta\left(f+\frac{1}{2\pi}\right) + \frac{1}{2i}\delta\left(f-\frac{1}{2\pi}\right)$$

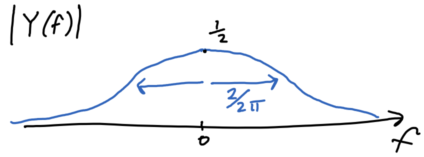


$$y(t) = u(t)e^{-2t}$$

$$Y(f) = \int_0^\infty e^{-2t}e^{-i2\pi ft}\,dt$$

$$= \int_0^\infty e^{(-2-i2\pi f)t}\,dt = \frac{1}{2+i2\pi f}$$

$|Y(f)|$



$$e^{-2\pi}\,y(t-\pi) = u(t-\pi)e^{-2t}$$

$$\Rightarrow \mathcal{F}(\ \downarrow\ ) = e^{-2\pi} e^{-i2\pi(\pi)f} Y(f)$$

$$= \frac{e^{-2\pi - i2\pi^2 f}}{2 + i2\pi f}$$

$$X(f) = (\quad) * \left(\frac{1}{2i}\delta\left(f - \frac{1}{2\pi}\right) + \frac{-1}{2i}\delta\left(f + \frac{1}{2\pi}\right)\right)$$

$$= \frac{e^{-2\pi}}{2i}\left(\frac{e^{-i2\pi^2\left(f - \frac{1}{2\pi}\right)}}{2 - i + i2\pi f} + \frac{e^{-i2\pi^2\left(f + \frac{1}{2\pi}\right)}}{2 + i + i2\pi f}\right)$$

From PS1:

$$x(t) = e^{-i\pi t} + e^{2(1 + i\pi t)}$$

$T = 2$ Period 2 Period 1

$$x(t) = \sum_k a_k e^{i2\pi \frac{k}{T} t}$$

$$= 1\, e^{i2\pi\left(\frac{-1}{2}\right)t} + e^2 e^{i2\pi\left(\frac{2}{2}\right)t}$$

a_{-1} a_2

CTFT: $X(f) = ?$ $\quad \delta\left(f + \frac{1}{2}\right) + e^2\delta(f - 1)$

1.) δ - functions all us to use the Fourier Transform on periodic signals.
2.) Must be done by inspections.

Time Domain / Frequency Domain		
	CTFS Period T $a_k = \frac{1}{T}\int_0^T x(t) e^{-i2\pi \frac{k}{T}t}dt$ $x(t) = \sum_k a_k e^{i2\pi\frac{k}{T}t}$	**DTFS** Period N $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-i2\pi\frac{k}{N}n}$ $x[n] = \sum_{k=0}^{N-1} a_k e^{i2\pi\frac{k}{N}n}$
	CTFT $X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t}dt$ $x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t}df$	**DTFT** $X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi f n}$ $x[n] = \int_{-1/2}^{1/2} X(f) e^{i2\pi f n}df$

← Periodic in time (finite power) Discrete

← Aperiodic Finite Energy Continuous

DFT ≅ DTFS

Finite duration discrete-time.

$$a_k = \sum_{n=0}^{N-1} x[n] e^{-i2\pi\frac{k}{N}n}$$

$x(t) = \int_{-\infty} X(f) e \; df \quad \Big| \quad x[n] = \int_{-1/2} X(f) e \; df$

Continuous-time — Aperiodic

Discrete-time — Periodic

Finite duration discrete-time.

$$a_k = \sum_{n=0}^{N-1} x[n] e^{-i2\pi \frac{k}{N} n}$$

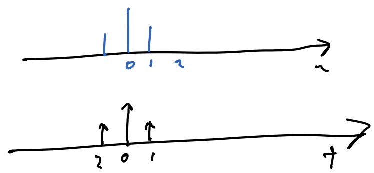
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} a_k e^{i2\pi \frac{k}{N} n}$$

DFT $\simeq$ DTFT

Represent DT as CT using δ-functions:

$x[n]$

Define $x(t) = \sum_n x[n]\,\delta(t-n)$



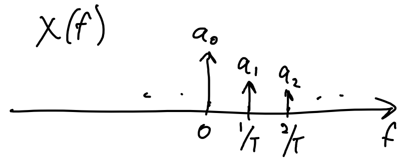
$$\mathcal{F}(x(t)) = \int_{-\infty}^{\infty} \Big(\sum_n x[n]\,\delta(t-n)\Big) e^{-i2\pi f t}\, dt$$

$$= \sum_n x[n] \int_{-\infty}^{\infty} \delta(t-n) e^{-i2\pi f t}\, dt$$

$$= \sum_n x[n]\, e^{-i2\pi f n} = \mathcal{F}(x[n])$$

Similarly: Suppose we have period T and $\{a_k\}$

Define $X(f) = \sum_k a_k\, \delta\!\left(f - \frac{k}{T}\right)$



$$\mathcal{F}^{-1}(X(f)) = \int_0^1 \Big(\sum_k a_k\, \delta\!\left(f-\tfrac{k}{T}\right)\Big) e^{i2\pi f t}\, df$$

$$= \sum_k a_k \int_{0-\varepsilon}^{1-\varepsilon} \delta\!\left(f-\tfrac{k}{T}\right) e^{i2\pi f t}\, df$$

$$= \sum_{k=0}^{N-1} a_k\, e^{i2\pi \frac{k}{T} t} = x[n]$$

System: $\boxed{y(t) = t^2 x(t-1)}$

Linearity: $a\,x(t) + b\,y(t) \longrightarrow t^2\big(a\,x(t-1) + b\,y(t-1)\big)$

$= a\,t^2 x(t-1) + b\,t^2 y(t-1)$

Yes

Time-invariance: $x(t-\tau) \longrightarrow t^2\, x((t-1)-\tau)$

$\neq (t-\tau)^2\, x((t-\tau)-1)$

No

Memoryless: No

Causal: Yes

Stable: No: Let $x(t) = 1 \Rightarrow y(t) = t^2$

Invertible: $x(t-1) = \frac{y(t)}{t^2} \Rightarrow x(t) = \frac{y(t+1)}{(t+1)^2}$ Not invertible at $t = -1$.

$e^{i2\pi Ft} \longrightarrow$ LTI H $\longrightarrow$

$e^{i2\pi Ft} \xrightarrow{\mathcal{F}} \delta(f-F)$

$H(f)\,\delta(f-F) = H(F)\,\delta(f-F)$

Convolution: $\int_{-\infty}^{\infty} h(\tau)\, e^{i2\pi F(t-\tau)}\, d\tau$

$= e^{i2\pi Ft} \int_{-\infty}^{\infty} h(\tau)\, e^{-i2\pi F\tau}\, d\tau$

$= e^{i2\pi Ft}\, \underline{H(F)}$

$\cos(2\pi Ft) \longrightarrow$ LTI H $\longrightarrow |H(F)| \cos\left(2\pi Ft + \measuredangle H(F)\right)$

What is the significance of $f = 0$?

FS: $a_0 = \frac{1}{T} \int_0^T x(t)\, dt \longleftarrow$ average. DC offset

Look at $X(0) = \int_{-\infty}^{\infty} x(t)\, dt$ $\longleftarrow$ DC Component

A δ-function at $f=0$ would cause an offset

Similar: $\underbrace{x(t) * x^*(-t)}_{\text{"Autocorrelation function"}} \Big|_{t=0}$

$\int_{-\infty}^{\infty} x(\tau)\, x^*(-(t-\tau))\, d\tau$ ($t=0$)

$= \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau)\, d\tau = E$

Invertibility is simple for LTI:

Impulse response $h(t)$.