ELE 201 Review

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Before we start reviewing, let me briefly remark on Hw4 which I graded (congratulations that most of you did a great job!)

1. Problem 2: without going into computations, the first thing to realize is that the answer must be a real symmetric function, since it is the Fourier transform of a real symmetric function. This kind of observation is helpful in verifying answers in the exams. **Exercise:** *An alternative way of solving this problem is to use the Fourier transform of rectangle function and then apply the sampling property. Comparing the results of the two methods, can you establish the identity* $\sum_{n\in\mathbb{Z}} \mathrm{sinc}(n) = \pi$*?*

2. Problem 4: the simplest method for this problem is based on Parseval's identity, although for those of you who didn't use the simplest argument, I also gave full credits in the homework if correct. **Exercise:** *Does the answer change if we further require the signal is casual?*

Now let's recall some key concepts related to Hw4. A system H is called linear if

$$H(ax + by) = aH(x) + bH(y) \tag{1}$$

for all inputs x, y and coefficients a, b. If in addition H commutes with the time shift operator, we say H is LTI. In this case, the matrix representation of H is a Toeplitz matrix, so it can be diagonalized by the Fourier basis; and the eigenvalues are the frequency response of the system. This is why Fourier analysis is so useful a tool for studying LTI systems.

As you saw from the problem set, the properties of Fourier transforms such as behaviors under scaling, inversion, convolution, multiplier, conjugation, taking energy... will be heavily tested, which I will not belabor here since you can find them in the textbooks.

As you saw in the labs, sampling/aliasing occurs in many practical applications. You can deduce the result of sampling from the behavior of Fourier transform under convolution. Suppose $g(t)$ is a sequence of pulses at intervals of T, i.e.

$$g(t) = \sum_{n\in\mathbb{Z}} \delta(t - nT). \tag{2}$$

Then for input signal $x(t)$, the sampled signal can be thought of as $x(t)g(t)$. The Fourier transform of the sampled signal is the convolution of $\hat{x}$ and $\hat{g}$. If we can show that

$$\hat{g}(f) = \frac{1}{T}\sum_{n\in\mathbb{Z}} \delta(f - \frac{n}{T}) \tag{3}$$

then $\hat{f} * \hat{g}$ must be the sum of translations of $\hat{f}$ with scaling factor of $1/T$, which recovers the sampling/aliasing property. However there is a slight difficulty in establishing (3): note that from (2) and the definition of Fourier transform we actually have

$$\hat{g}(f) = \sum_{n\in\mathbb{Z}} e^{-2\pi i f n T}. \tag{4}$$

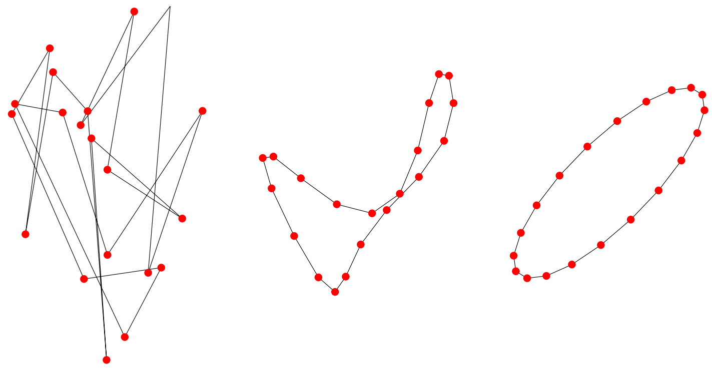

FIG. 1.4. *The progression from $\mathcal{P}_0$ to $\mathcal{P}_{18}$ to $\mathcal{P}_{200}$ for an $n = 20$ example.*

which looks different from (3). But thinking about the cancellations going on in (4), we realize that it is equivalent to the following form

$$\hat{g}(f) = c \sum_{n \in \mathbb{Z}} \delta(f - \frac{n}{T}) \tag{5}$$

where c is some constant. When I was an undergrad I used to worry about forgetting the value of c. Luckily, there is a number of tricks to see at least that c is inversely proportional to T, such as scaling property of Fourier transform, or observing that $\hat{g}(f)$ is a unit-less number (also known as dimensional analysis). To determine the exact value of c, one can integrate (4) and (5) over $[0, \frac{1}{T})$, which gives $\frac{1}{T} = c$. This example illustrates some tricks to get the correct answer quickly in the exam without being rigorous.

Finally I would like to mention a fun problem which illustrates how LTI system theory can be applied.

> Suppose $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are n-vectors, and let $P_0(\mathbf{x}, \mathbf{y})$ be the polygon obtained by connecting the points $(x_1, y_1), \ldots, (x_n, y_n), (x_1, y_1)$ on the plane in that order. Let $P_1(\mathbf{x}, \mathbf{y})$ be the n-gon formed by connecting the midpoints of the neighboring edges of $P_0(\mathbf{x}, \mathbf{y})$, and iterate this process... what does $P_k(\mathbf{x}, \mathbf{y})$ look like for large k? (Yes it converges to the center of mass eventually. But what if you zoom in with a factor of $\cos^{-k}(\pi/n)$?)

The figure is excerpted from [1], which analyzed the problem based on the eigenvalues of the linear operator involved. Can you come up with a simple analysis using the property of filtering we learnt in class? Can you generalize the result to higher dimensions?

That's all folks and good luck with your exams!

References

[1] Adam N Elmachtoub and Charles F Van Loan. From random polygon to ellipse: an eigenanalysis. *SIAM review*, 52(1):151–170, 2010.