Lecture 10

Friday, February 27, 2015 8:43 AM

Convolution:

DT : $(x*y)[n] = \sum_{k=-\infty}^{\infty} x[k]\, y[n-k]$

CT : $(x*y)(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

Notation: $(x*y)(t)$ often written as $\underline{x(t)*y(t)}$

- Reason this is convenient: Concisely write convolutions of manipulated signals.

  e.g. $x(t-T)*y(3t)$

  Without this notation:

  $x_2(t) = x(t-T)$

  $y_2(t) = y(3t)$

  $(x_2 * y_2)(t)$

- Reason this is confusing: Convolution is a function of the entire signal for each output time t.

http://mathworld.wolfram.com/Convolution.html

Continous-time Delta Function (Dirac delta function)

DT : $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$ $\qquad 1_{\{0\}}[n]$

CT : $\delta(t)$ is not actually a function

Define : $\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$ if f is continuous at $t=0$.

Sifting property

Sifting Prop.: $\int_{-\infty}^{\infty} \delta(t-T) f(t)\, dt = f(T)$ if $f(t)$ is cont. at $t=T$.

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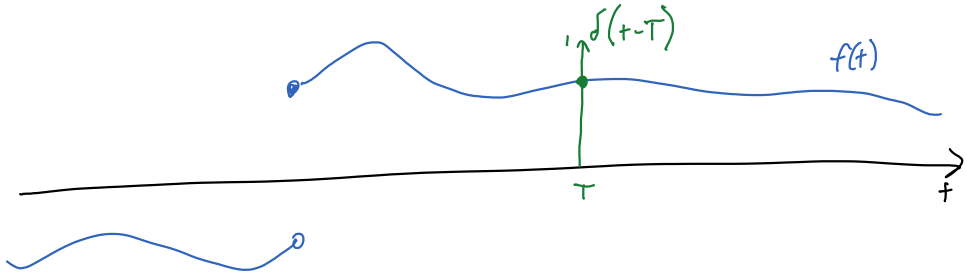
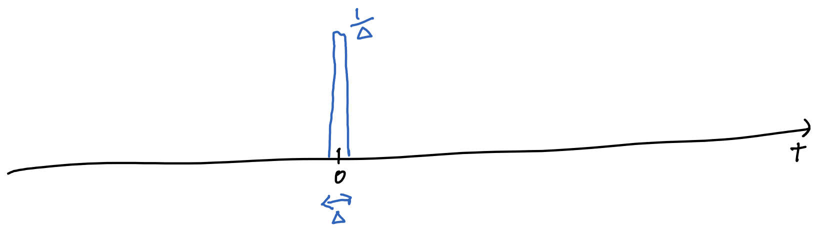
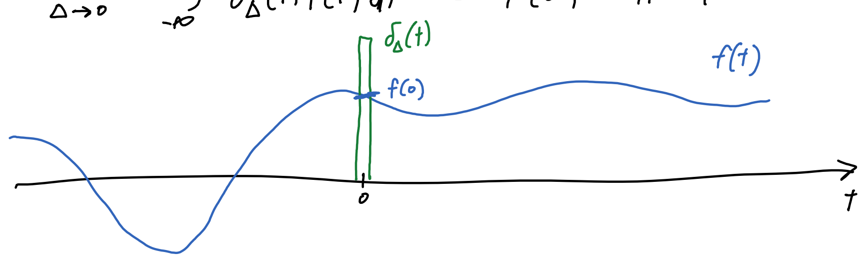


Delta function like a short pulse: $\delta_\Delta(t) \triangleq \dfrac{\text{rect}(t/\Delta)}{\Delta}$



This behaves like $\delta(t)$ for small Δ.

Precisely: $\displaystyle\lim_{\Delta\to 0} \int_{-\infty}^{\infty} \delta_\Delta(t) f(t)\, dt = f(0)$ if f cont. at $t=0$.



Product: $\delta_\Delta(t) f(t) \approx \dfrac{\text{rect}(t/\Delta)}{\Delta} f(0)$

Tempting to say $\displaystyle\lim_{\Delta\to 0} \delta_\Delta(t) = \delta(t)$

Not true: $\displaystyle\lim_{\Delta\to 0} \delta_\Delta(t) = \begin{cases} 0, & \forall t \neq 0 \\ \infty, & t=0 \end{cases}$

More importantly, the functions do not converge in L_2.

Convenient substitutions: 1.) $f(t)\,\delta(t-T) = f(T)\,\delta(t-T)$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-T)\,dt = f(T) = \int_{-\infty}^{\infty} f(T)\,\delta(t-T)\,dt$$

$$= f(T)\int_{-\infty}^{\infty} \delta(t-T)\,dt$$

Delta function has integral one: $\int_{-\infty}^{\infty} \delta(t-T)\,dt = \int_{-\infty}^{\infty} 1\,\delta(t-T)\,dt = 1$

2.) $\delta(at) = \frac{1}{|a|}\delta(t)$

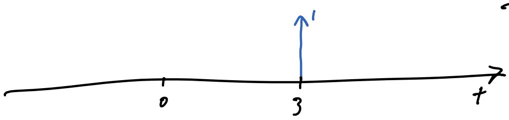
Check: $\int_{-\infty}^{\infty} \delta(at)\,dt$ Let $\tau = at$

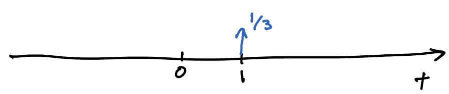
$= \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(\tau)\,d\tau$

$= \frac{1}{|a|}$

$x(t) = \delta(t-3)$ Scale horizontally $x(3t) = \delta(3t-3) = \delta(3(t-1))$

$= \frac{1}{3}\delta(t-1)$





How to interpret magnitude and phase of frequency components:

Magnitude = "amplitude"

phase is time shift

Recall Fourier Series : $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{i 2\pi \frac{k}{T} t}$

If $x(t)$ is real, $a_k = a_{-k}^*$

$$\Rightarrow \quad x(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{i2\pi \frac{k}{T} t} + a_k^* e^{-i2\pi \frac{k}{T} t}$$

$$= a_0 + \sum_{k=1}^{\infty} |a_k| \cos\left(2\pi \frac{k}{T} t + \measuredangle a_k\right)$$

Fourier Transform Derivative Property

$$x(t) \xrightarrow{\mathcal{F}} X(f)$$

$$\frac{d}{dt} x(t) \xrightarrow{\mathcal{F}} i2\pi f \, X(f)$$