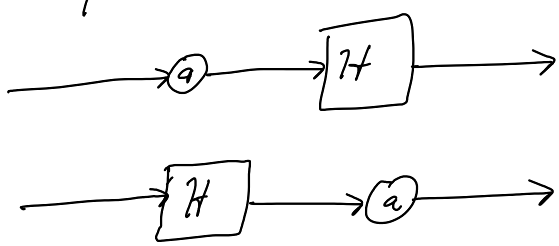
Lecture 12

Friday, March 06, 2015 9:00 AM

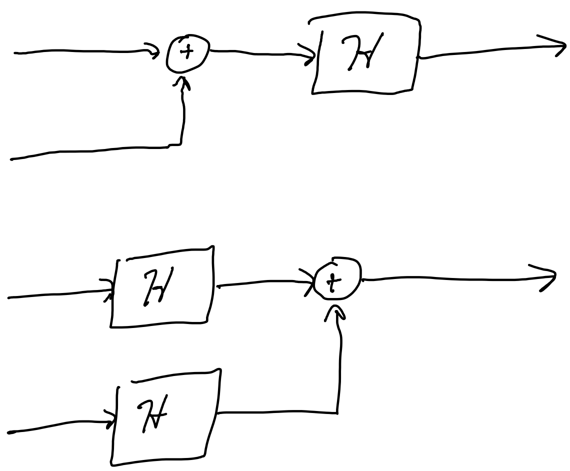
Observations using block diagrams:

Linearity: System commutes with sums and scaling.

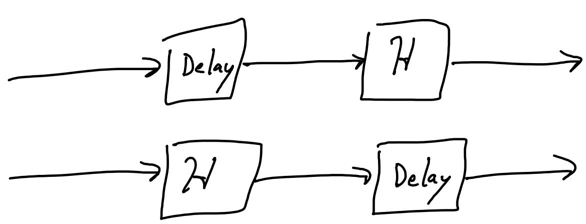
1.)



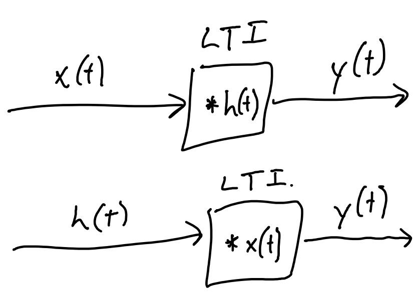
1.)



Time-invariance: System commutes with time delay (or advance)



We know convolution commutes: $x(t) * h(t) = h(t) * x(t)$



$$\underbrace{\delta(t)}_{\text{input}} * \underbrace{h(t)}_{\text{impulse response}} = h(t), \quad \text{as we've seen already}$$

Exercise: What is $\delta(t-3) * h(t-3)$?

$h(t-6)$

$$\int_{-\infty}^{\infty} \delta(\tau-3)\, h((t-\tau)-3)\, d\tau \qquad \tau = 3 \quad \text{sifting}$$

$$= h((t-3)-3) = h(t-6)$$

System is LTI and ...

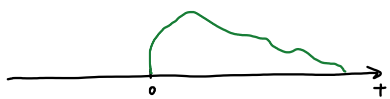
Memoryless: Must be an amplifier: $y(t) = a\,x(t)$

$$\boxed{h(t) = a\,\delta(t)}$$

Example of not-LTI but memoryless system:

$$y(t) = |x(t)|^{+}$$

Causal: $h(t) = 0 \quad \forall t < 0.$



Stable: $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$ ← "Absolutely integrible"

Proof: $\int_{-\infty}^{\infty} |h(t)|dt < \infty \Rightarrow$ Stable

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \right|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)\, x(t-\tau)|\, d\tau$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)|\, B_x\, d\tau$$

$$= B_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \quad \leftarrow \text{Bound } B_y$$

Prove Stable $\Rightarrow \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

by contradiction. Assume $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$

Let $x(t) = \text{sign}(h(-t))$ where $\text{sign}(z) = \begin{cases} 1, & z > 0 \\ -1, & z \leq 0 \end{cases}$

Notice $|x(t)| \leq 1 \quad \forall t$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

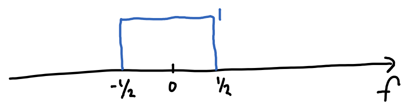
$$y(0) = \int_{-\infty}^{\infty} h(\tau)\, x(-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \text{sign}(h(\tau))\, d\tau$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \infty$$

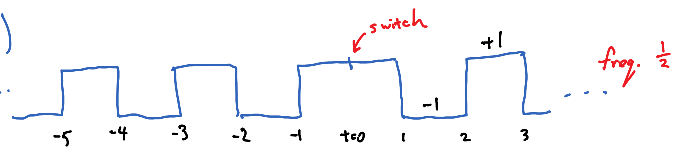
Example of unstable LTI system:

Ideal LPF

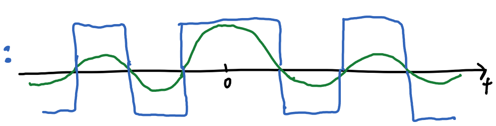


$h(t) = \text{sinc}(t)$

Let the input be



Output at time zero is integral of product:



$$= \int_{-\infty}^{\infty} |\text{sinc}(t)|\, dt = \infty$$

Invertible: Something to think about after this lecture.

Connection to the Fourier Transform:

Let $y(t) = x(t) * h(t)$ and $x(t) \xrightarrow{\mathcal{F}} X(f)$

$h(t) \xrightarrow{\mathcal{F}} H(f)$

$$Y(f) = \mathcal{F}(y(t)) = \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi f t}\, dt$$

$$= \int \int_{\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \;\; e^{-i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \;\; e^{-i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)\, e^{-i2\pi f t}\, dt \right) d\tau \qquad \tau' = t - \tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(\tau')\, e^{-i2\pi f(\tau+\tau')}\, d\tau' \right) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f \tau} \left(\int_{-\infty}^{\infty} h(\tau')\, e^{-i2\pi f \tau'}\, d\tau' \right) d\tau$$

$$= \left(\int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f \tau}\, d\tau \right) \left(\int_{-\infty}^{\infty} h(\tau')\, e^{-i2\pi f \tau'}\, d\tau' \right)$$

$$= X(f)\, H(f)$$

Convolution Property!

$$x(t) * h(t) \xrightarrow{\mathcal{F}} X(f)\, H(f)$$

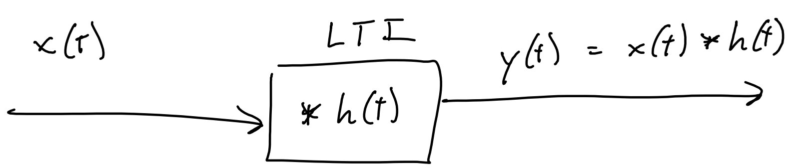
i.e. $\mathcal{F}(x(t) * h(t)) = \mathcal{F}(x(t)) \cdot \mathcal{F}(h(t))$

Same property for D.T.

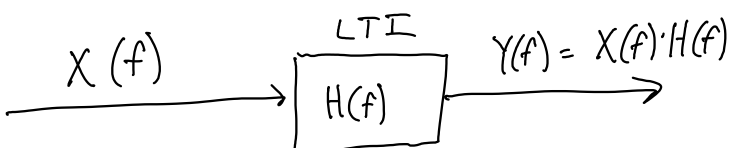
Most important feature of Fourier transform

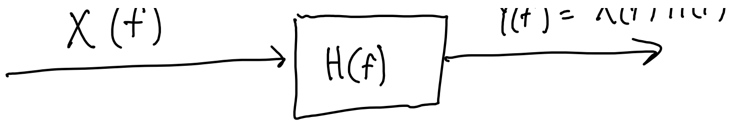
"Diagonalizes" LTI system

Concept from linear algebra



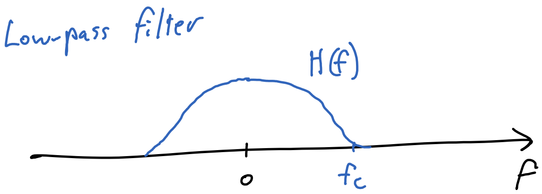
Simple to think about frequency domain:





Filtering: Design an LTI system $\mathcal{H}$ to have a desired effect.
Best interpretted in frequency domain

Low-pass filter



To implement this filter

$$h(t) = \mathcal{F}^{-1}\left(\mathcal{H}(f)\right)$$

Specific example: Local average for smoothing:

$$y(t) = \frac{1}{W}\int_{-W/2}^{-W/2} x(t-\tau)\,d\tau$$

Local averaging is LTI

$$y(t) = x(t) * h(t)$$

What is $h(t)$?

Inpection: $h(t) = \frac{1}{W}\,\mathrm{rect}(t/W)$

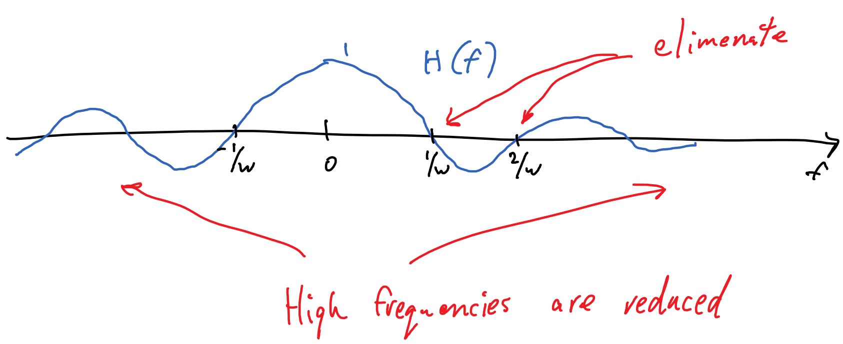
Plug in: $h(t) = \frac{1}{W}\int_{-W/2}^{W/2} \delta(t-\tau)\,d\tau$

$$= \frac{1}{W}\,\mathrm{rect}(t/W)$$

What does this do in freq. domain?

$$H(f) = \mathcal{F}\left(h(t)\right) = \frac{1}{W}\,\mathcal{F}\left(\mathrm{rect}(t/W)\right)$$

$$= \mathrm{sinc}(Wf)$$



Sometimes it is faster to compute $\mathcal{F}^{-1}\left(\mathcal{F}(x(t)) \cdot \mathcal{F}(h(t))\right)$ than $x(t) * h(t)$.

This can be true on paper or for real signals on computer (using FFT).