Lecture 13

Monday, March 09, 2015 8:18 AM

DFT → DTFS

Time Domain / Freq. Domain	Continuous	Discrete Periodic
Periodic Discrete (Finite Power)	**CTFS** Period T: $a_k = \frac{1}{T}\int_0^T x(t) e^{-i2\pi \frac{k}{T} t} dt$, $x(t) = \sum_k a_k e^{i2\pi \frac{k}{T} t}$	**DTFS** Period N: $a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-i2\pi \frac{k}{N} n}$, $x[n] = \sum_{k=0}^{N-1} a_k e^{i2\pi \frac{k}{N} n}$
Continuous (Finite Energy)	**CTFT**: $X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$, $x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df$	**DTFT**: $X(f) = \sum_n x[n] e^{-i2\pi f n}$, $x[n] = \int_{-1/2}^{1/2} X(f) e^{i2\pi f n} df$

Most general.

Delta Function Properties (continued)

What about integrating over a restricted set?

$$\int_A \delta(t-t_0) f(t) dt = \int_{-\infty}^{\infty} \delta(t-t_0)\left(f(t)\, 1_A(t)\right) dt$$

$$= f(t_0)\, 1_A(t_0)$$

$$= \begin{cases} f(t_0) & \text{if } t_0 \in A \\ 0 & \text{if } t_0 \notin A \end{cases}$$

Delta functions in the integration interval are all that matter.

What is $\int_{-\infty}^{t} \delta(\tau) d\tau = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} = u(t)$

$\delta(t) = u'(t)$

What is the energy of $\delta(t)$?

$$\int_{-\infty}^{\infty} \delta^2(t)\, dt = \infty$$

Consider rect. approximation:

$$E\left\{\delta_\Delta(t)\right\} = \frac{1}{\Delta} \longrightarrow \infty \text{ as } \Delta \to 0.$$

$$\int_{-\infty}^{\infty} \delta(t)\, \delta(t)\, dt$$

$$\delta(t-t_0) * x(t) = x(t-t_0)$$

$$\delta(t-t_0) * \left(3\delta(t-1)\right) = 3\,\delta(t-t_0-1)$$

Fourier Transform with $\delta(t)$:

$$\delta(t-t_0) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i2\pi f t}\, dt = e^{-i2\pi t_0 f}$$

$$\int_{-\infty}^{\infty} \delta(f-f_0)\, e^{i2\pi f t}\, df = e^{i2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f-f_0)$$

$$\mathcal{F}\left(\sin(t)\right) = \mathcal{F}\left(\frac{1}{2i}\left(e^{it} - e^{-it}\right)\right)$$

$$= \mathcal{F}\left(\frac{1}{2i} e^{i2\pi\left(\frac{1}{2\pi}\right)t} - \frac{1}{2i} e^{-i2\pi\left(\frac{1}{2\pi}\right)t}\right)$$

$$= \frac{1}{2i}\, \delta\left(f - \frac{1}{2\pi}\right) - \frac{1}{2i}\, \delta\left(f + \frac{1}{2\pi}\right)$$

Example from PS 1:

$$x(t) = e^{-i\pi t} + e^{2(1+i\pi t)}$$

Period 2   Period 1

$$x(t) = \sum_k a_k e^{i2\pi \frac{k}{T} t}$$

$T=2$

$$= 1\, e^{i2\pi\left(\frac{-1}{2}\right)t} + e^{2}\, e^{i2\pi\left(\frac{2}{2}\right)t}$$

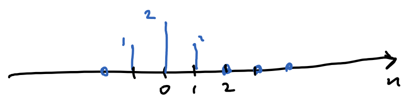
$a_{-1} = 1$ $a_2 = e^2$

CTFT: $X(f) = ?$ $\qquad \delta(f+\frac{1}{2}) + e^2\delta(f-1)$

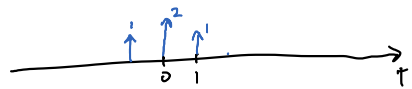
1.) δ-functions allow us to use FT on periodic signals.
2.) Must be done by inspection.

Represent DT as CT using δ-functions:

$x[n]$



Define $x(t) = \sum_n x[n]\,\delta(t-n)$

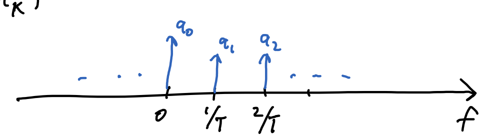


$$\mathcal{F}(x(t)) = \int_{-\infty}^{\infty}\left(\sum_n x[n]\,\delta(t-n)\right)e^{-i2\pi ft}\,dt$$

$$= \sum_n x[n]\int_{-\infty}^{\infty}\delta(t-n)\,e^{-i2\pi ft}\,dt$$

$$= \sum_n x[n]\,e^{-i2\pi fn} \quad = \quad \mathcal{F}(x[n])$$

Similarly: Suppose we have period T and $\{a_k\}$



Define $X(f) = \sum_k a_k\,\delta(f-\frac{k}{T})$

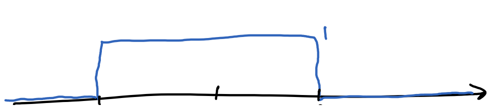
$$\mathcal{F}^{-1}(X(f)) = \int_{-\infty}^{\infty}\left(\sum_k a_k\,\delta(f-\tfrac{k}{T})\right)e^{i2\pi ft}\,df$$

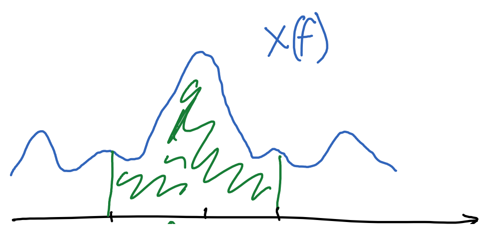
$$= \sum_k a_k\int_{-\infty}^{\infty}\delta(f-\tfrac{k}{T})\,e^{i2\pi ft}\,df$$

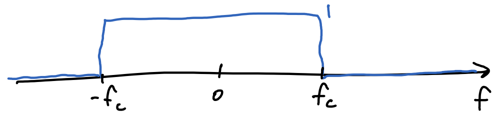
$$= \sum_k a_k\,e^{i2\pi\frac{k}{T}T} \quad = \quad x(t)$$

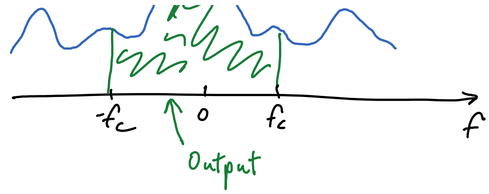
Ideal Low-Pass filter:

$H(f) = \text{rect}(f/2f_c)$



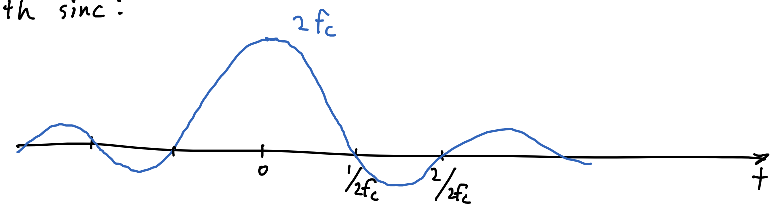






Time-domain: $h(t) = \mathcal{F}^{-1}\left(\text{rect}\left(f/2f_c\right)\right) = 2f_c \,\text{sinc}(2f_c t)$

Convolution with sinc!



Think of $h(t)$ as weights for averaging.

Ideal LPF not always the best smoother.

Not very localized in time.

Inputs affect the output for a long time.

"Ringing"