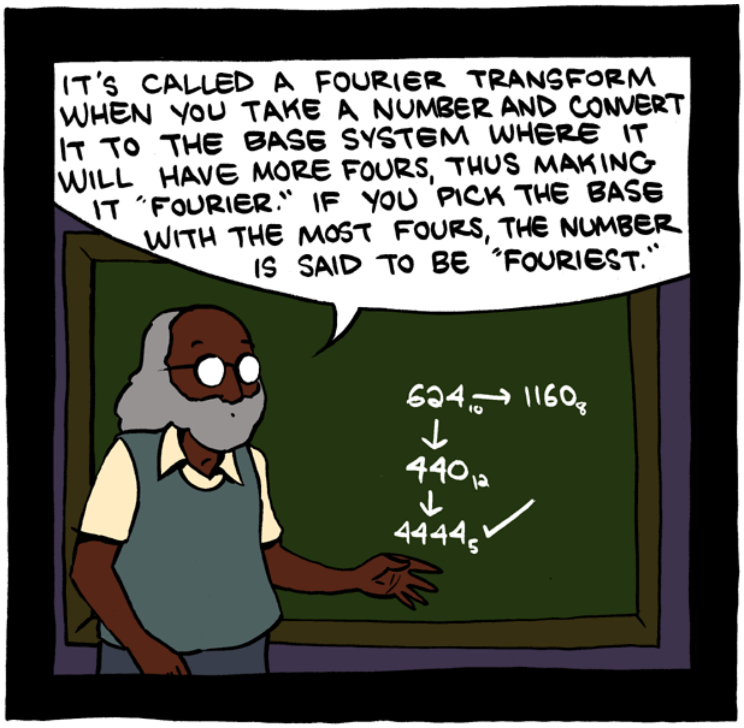
Lecture 14

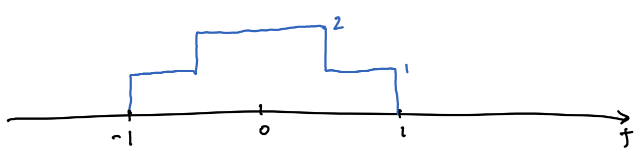
Monday, March 09, 2015 6:01 PM



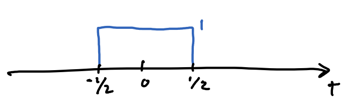
Teaching math was way more fun after tenure.

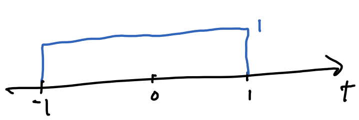
Properties of the Fourier Transform:

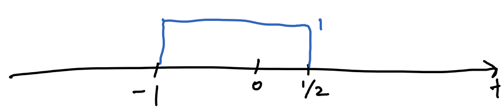
Linearity Example:

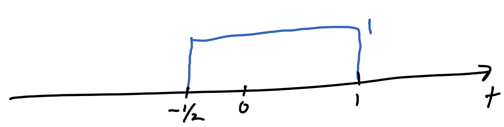


Decompose into rects









$x(t) = \text{rect}(t/2) + \text{rect}(t)$

$X(f) = 2\,\text{sinc}(2f) + \text{sinc}(f)$

or

$x(t) = \text{rect}\left(\frac{2}{3}\left(t+\frac{1}{4}\right)\right) + \text{rect}\left(\frac{2}{3}\left(t-\frac{1}{4}\right)\right)$

scale then shift

$$X(f) = 2\,\mathrm{sinc}(2f) + \mathrm{sinc}(f)$$

Surprisingly, the same

scale then shift

$$\mathrm{rect}(t) \longrightarrow \mathrm{sinc}(f)$$

$$\mathrm{rect}\left(\tfrac{2}{3}t\right) \longrightarrow \tfrac{3}{2}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right)$$

$$\mathrm{rect}\left(\tfrac{2}{3}\left(t+\tfrac{1}{4}\right)\right) \longrightarrow e^{i\frac{\pi}{2}f}\,\tfrac{3}{2}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right)$$

$$\Rightarrow X(f) = \tfrac{3}{2}e^{i\frac{\pi}{2}f}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right) + \tfrac{3}{2}e^{-i\frac{\pi}{2}f}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right)$$

Euler:

$$\tfrac{3}{2}e^{i\frac{\pi}{2}f}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right) + \tfrac{3}{2}e^{-i\frac{\pi}{2}f}\,\mathrm{sinc}\left(\tfrac{3}{2}f\right) = \frac{e^{i\frac{\pi}{2}f}\sin\left(\tfrac{3}{2}\pi f\right) + e^{-i\pi/2 f}\sin\left(\tfrac{3}{2}\pi f\right)}{\pi f}$$

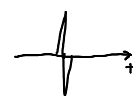
$$= 2\,\frac{\cos\left(\tfrac{\pi}{2}f\right)\sin\left(\tfrac{3}{2}\pi f\right)}{\pi f} = \frac{\sin(\pi f) + \sin(2\pi f)}{\pi f}$$

Derivative Property:

$$\frac{d}{dt}x(t) \xrightarrow{\mathcal{F}} i2\pi f\, X(f)$$

Amplify higher frequencies

LTI process: $h(t) = \delta'(t)$



$\Rightarrow$ integration is opposite.

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{F}} \frac{1}{i2\pi f}X(f) + \frac{X(0)}{2}\delta(f)$$

What about DC offset $(f=0)$? or else

This assumes $\int_{-\infty}^{\infty} x(t)\,dt = 0$.

Proof:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{i2\pi ft}\,df$$

$$= \int_{-\infty}^{\infty} X(f)\left(\frac{d}{dt}e^{i2\pi ft}\right)df$$

$$= \int_{-\infty}^{\infty} i2\pi f\,X(f)\,e^{i2\pi ft}\,df$$

$$= \mathcal{F}^{-1}\left(i2\pi f\,X(f)\right) \quad \square$$

Alternate derivation:

Sifting prop. for $\delta'(t)$:

$$\int_{-\infty}^{\infty}\delta'(t-T)f(t)\,dt = -f'(T)$$

$$\Rightarrow \mathcal{F}(h(t)) = \mathcal{F}(\delta'(t)) = -(-i2\pi f)e^{0}$$

DT Difference property:

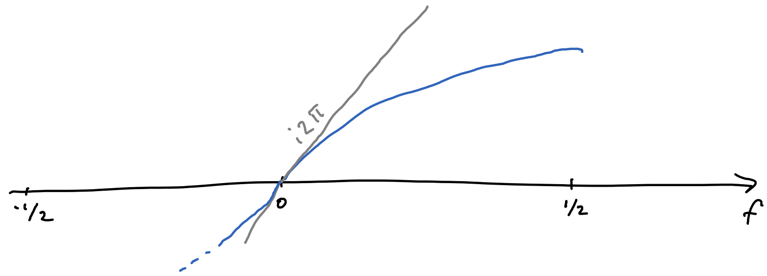
$$x[n] - x[n-1] \xrightarrow{\mathcal{F}} X(f) - e^{-i2\pi f}X(f)$$

$$= \left(1 - e^{-i2\pi f}\right)X(f)$$

$$= e^{-i\pi f}\left(e^{i\pi f} - e^{-i\pi f}\right) X(f)$$

$$= i2\, e^{-i\pi f}\, \sin(\pi f)\, X(f)$$

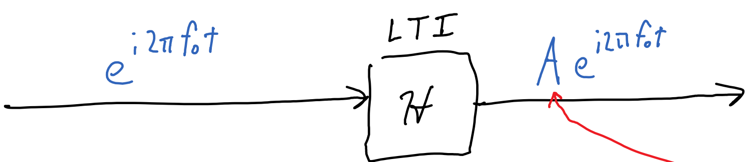
For small f, $e^{-i\pi f} \approx 1$, $\sin(\pi f) \approx \pi f$



Transfer Function of a system:

(another look at the convolution property)

Complex exponentials are the "eigensignals" for any LTI system.



<u>Proof</u>: Convolution:

$$y(t) = x(t) * h(t)$$

$$= h(t) * x(t)$$

$$= h(t) * e^{i2\pi f_0 t}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{i2\pi f_0 (t-\tau)}\, d\tau$$

$$= e^{i2\pi f_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-i2\pi f_0 \tau}\, d\tau$$

This number is the eigenvalue

$H(f_0)$

We could do this derivation without ever having introduced the Fourier transform

Another explanation (derivation):

$$X(f) = \mathcal{F}\left(e^{i2\pi f_0 t}\right) = \delta(f - f_0)$$

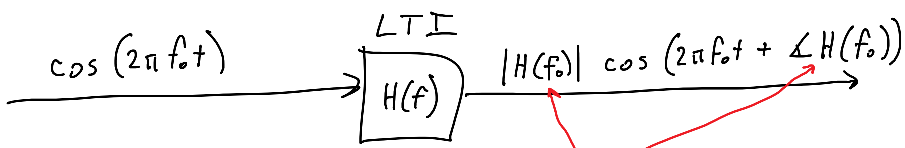
$$X(f) = \mathcal{F}\left(e^{i2\pi f_0 t}\right) = \delta(f - f_0)$$

$$Y(f) = X(f)\,H(f) = \delta(f-f_0)\,H(f) = \delta(f-f_0)\,H(f_0)$$

$$y(t) = \mathcal{F}^{-1}\left(\delta(f-f_0)\,H(f_0)\right) = H(f_0)\,\mathcal{F}^{-1}\left(\delta(f-f_0)\right)$$

$$= H(f_0)\,e^{i2\pi f_0 t}$$

What about a real sinusoid?



This is why we usually represent $H(f)$ in terms of magnitude and phase.

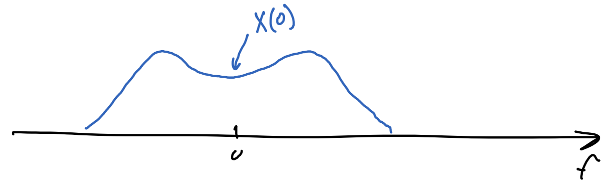
DC component:

(DC offset
DC bias)

Consider $f = 0$.

Fourier Series: $a_0 = \frac{1}{T}\int_0^T x(t)\,dt$ ← average DC offset

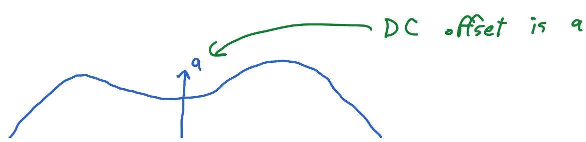
Fourier Transform: $X(0) = \int_{-\infty}^{\infty} x(t)\,dt$ ← DC component

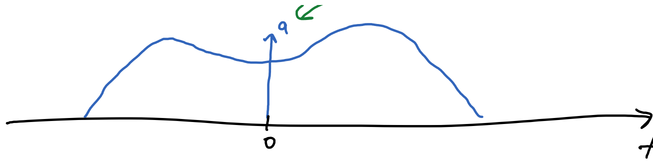


Consider a δ-function at $f=0$ (or $t=0$):

$$\delta(t) \xrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t)\,e^{-i2\pi f t}\,dt = e^{-i2\pi f(0)} = 1$$ ← All frequencies

$$1 \xrightarrow{\mathcal{F}} \delta(f)$$ ← DC offset





A useful trick: Evaluate $\int_{-\infty}^{\infty} f(t)\,dt$ as $F(0)$ (e.g. $\int_{-\infty}^{\infty} \operatorname{sinc}(t)\,dt = \operatorname{rect}(0) = 1$)

Similar trick: $r(t) = x(t) * x^*(-t)$ is called the "autocorrelation function"

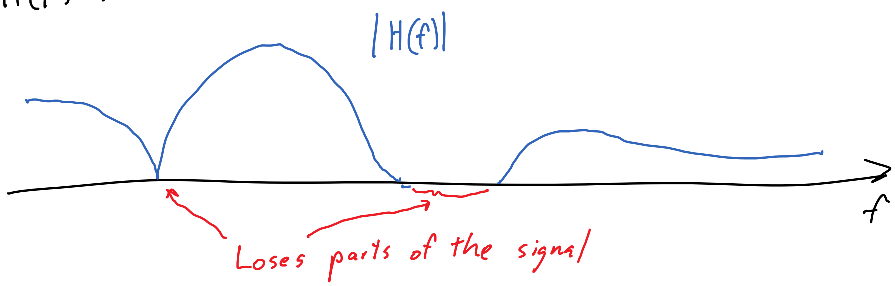
Notice that $r(0) = \text{Energy}$

Proof:
$$r(t) = \int_{-\infty}^{\infty} x(\tau)\, x^*(-(t-\tau))\,d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau - t)\,d\tau$$
$$\Rightarrow r(0) = \int_{-\infty}^{\infty} x(\tau)\, x^*(\tau)\,d\tau = E$$

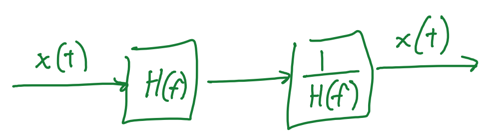
LTI and invertibility:

Easy to check:

Let $H(f)$ be the transfer function (i.e. $\mathcal{F}(h(t))$)



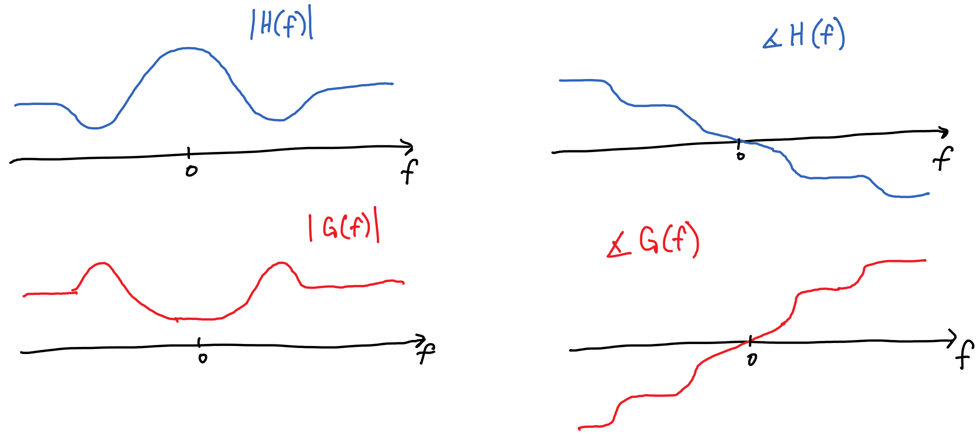
If $H(f) \neq 0 \;\forall f \Rightarrow$ invertible.



If $H(f)$ crosses zero only at isolated points then almost invertible.
Only loses pure tones (δ-functions in frequency)

Example of inverse:

$|H(f)|$ $\measuredangle H(f)$



$H(f) \cdot G(f) = 1 \quad \Rightarrow \quad G$ is the inverse system of H.

i.e. $x(t) \rightarrow \boxed{H} \rightarrow \boxed{G} \rightarrow x(t)$

Notice: $|G(f)| = \dfrac{1}{|H(f)|}$ and $\measuredangle G(f) = -\measuredangle H(f)$