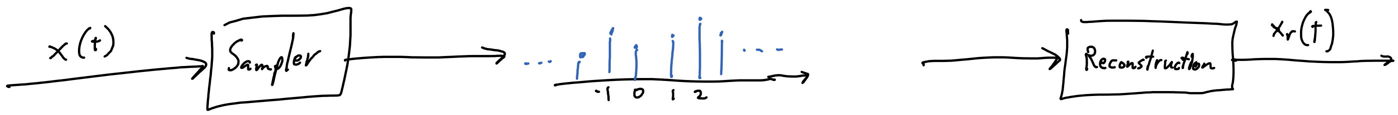
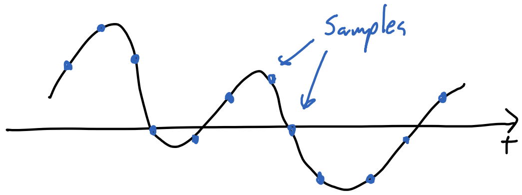
Sampling



Can we make $x_r(t) = x(t)$?

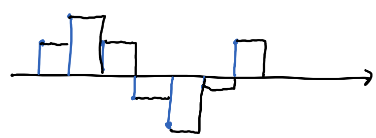
Ideal Uniform Sampler:



$$y[n] = x(nT_s)$$

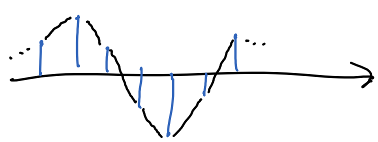
Reconstruction ideas:

Zero-order hold:



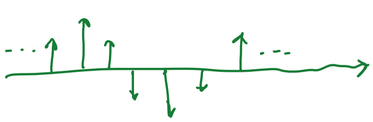
Easy to implement.
("sample and hold" circuit)

First-order hold:

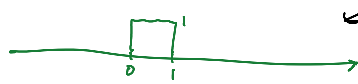


Notice these are both convolutions:

- Represent sample in cont.-time using δ-function

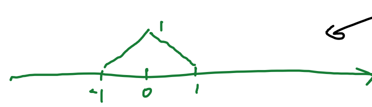


$*$



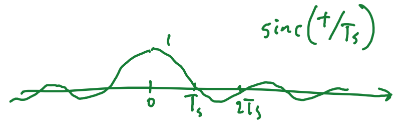
Zero-order hold

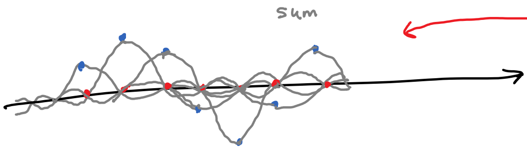
$*$



First-order hold

Sinc interpolation: $\longrightarrow$ $*$





Exact recovery under certain assumptions

Easy to verify that this is an interpolation

Fourier Transform Properties:

Parseval's Theorem:

(General property for norms and orthonormal transforms)

CTFT

$$E(x(t)) = E(X(f))$$

ie. Energy can be computed in time or frequency

$$\int |x(t)|^2 dt = \int |X(f)|^2 df$$

DTFT:

$$E(x[n]) = P(X(f))$$

Power: Integrate over only one period

CTFS:

$$P(x(t)) = E(\{a_k\}) = \sum_k |a_k|^2$$

Proof:

$$x(t) = \sum_k a_k \underbrace{e^{i 2\pi \frac{k}{T} t}}_{\text{orthonormal}}$$

$$P(x(t)) = \sum_k P(a_k \underbrace{e^{i2\pi \frac{k}{T} t}}_{\text{magnitude } 1}) = \sum_k |a_k|^2$$

DTFS:

$$P(x[n]) = N\, P(\{a_k\}) = \sum_{k=0}^{N-1} |a_k|^2$$

DFT:

$$N\, E(x[n]) = E(DFT(x[n]))$$

Real Part, Imag Part:
(Even, Odd)

Real Part, Imag Part: (Even, Odd)

$x(t)$ even and real $\Rightarrow$ $\mathcal{F}(x(t))$ even and real

even and imag $\Rightarrow$ even and imag

odd and real $\Rightarrow$ odd and imag

odd and imag $\Rightarrow$ odd and real

Check: let $x(t)$ be odd and real

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt$$

$$\Rightarrow X(-f) = \int_{-\infty}^{\infty} x(t)\, e^{i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(-\tau)\, e^{-i2\pi f \tau}\, d\tau$$

$$= -\int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi f \tau}\, d\tau = -X(f) \quad \Rightarrow \underline{\text{odd}}$$

$$X^*(f) = \left(\int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt \right)^*$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{i2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{i2\pi f t}\, dt = -X(f) \quad \Rightarrow \text{imag.}$$

Implications:

real-even part etc.

$$x(t) = x_{re}(t) + x_{ro}(t) + x_{ie}(t) + x_{io}(t)$$

Time-domain: Frequency Domain:

$x_{re}(t) \rightarrow X_{re}(f)$

$x_{ro}(t) \rightarrow X_{io}(f)$

$x_{io}(t) \rightarrow X_{ro}(f)$

$x_{io}(t) \longrightarrow X_{io}(f)$

$x_{ie}(t) \longrightarrow X_{ie}(f)$

Suppose $x(t)$ is real:

then $X(f)$ has a real-even part and imag-odd part.

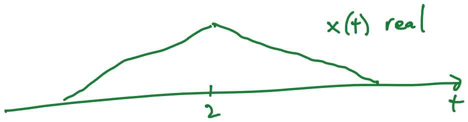
i.e. $\quad re(X(f))$ is even

$\qquad im(X(f))$ is odd

and $\quad |X(f)|$ is even

$\qquad \angle X(f)$ is odd

What if $x(t)$ has symmetry:



Not even

but $\hat{x}(t) = x(t+2)$ is even and real

$\Rightarrow x(t) = \hat{x}(t-2)$

$\hat{X}(f)$ is even and real (zero phase)

$$X(f) = e^{-i4\pi f}\hat{X}(f)$$

$\angle X(f) = -4\pi f$ (linear phase)

Conjugation:

$$x^*(t) \xrightarrow{\mathcal{F}} X^*(-f)$$

Multiplication Property:
(Modulation)

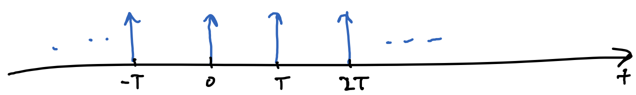
$$x(t)\cdot y(t) \xrightarrow{\mathcal{F}} X(f) * Y(f)$$

i.e. $\mathcal{F}\left(x(t)\cdot y(t)\right) = \mathcal{F}\left(x(t)\right) * \mathcal{F}\left(y(t)\right)$

Self-similar F.T. pairs:

Impulse Train: $\quad Ш_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

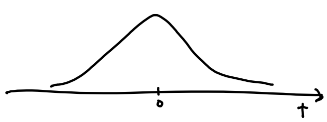
(Delta Train
Dirac Comb)



$$Ш_1(t) \xrightarrow{\mathcal{F}} Ш_1(f)$$

$$Ш_{T_s}(t) \xrightarrow{\mathcal{F}} \frac{1}{T_s}\, Ш_{1/T_s}(f)$$ ← Use scaling prop. of F.T. and scaling prop. of δ-func.

Gaussian: $\quad e^{-\pi t^2} \xrightarrow{\mathcal{F}}$ same



sinc(t) + rect(t) $\xrightarrow{\mathcal{F}}$ same