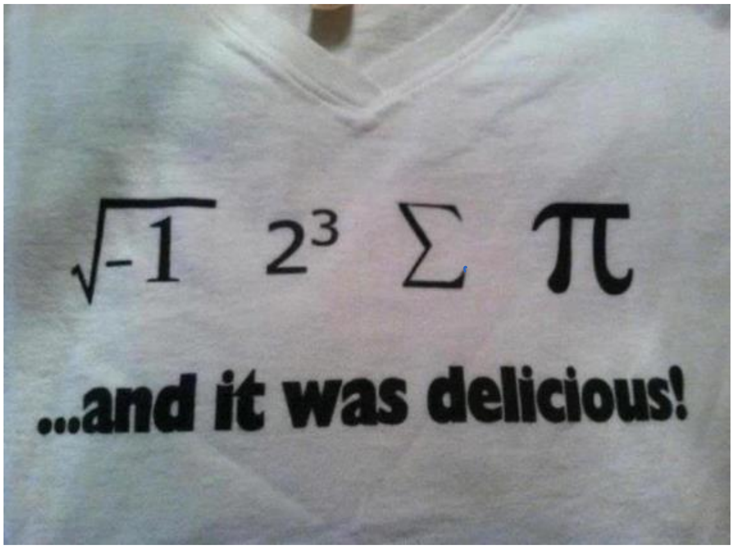
Lecture 17

Wednesday, March 25, 2015 10:26 AM

11:32 PM

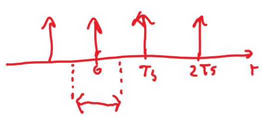


Sampling:

$$\boxed{\text{Verify that } \underbrace{\sum_k \delta(t-kT_s)}_{Ш_{T_s}(t)} \xrightarrow{\mathcal{F}} \underbrace{\frac{1}{T_s}\sum_k \delta\left(f-\frac{k}{T_s}\right)}_{\frac{1}{T_s}Ш_{\frac{1}{T_s}}(f)} \text{ using the Fourier Series.}}$$

$x(t) = \sum_k \delta(t-kT_s)$ has period T_s.

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \sum_k \delta(t-kT_s)\, e^{-i2\pi \frac{k}{T_s}t}\,dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-i2\pi\frac{k}{T_s}t}\,dt$$



Only one delta function is in the interval.

$$= \frac{1}{T_s}e^{-i2\pi\frac{k}{T_s}(0)} = \frac{1}{T_s} \quad \forall k$$

Comment: We already know this.

$$\delta(t) \xrightarrow{\mathcal{F}} 1$$

Convert CTFS to CTFT:

$$X(f) = \sum_k a_k\, \delta\left(f-\frac{k}{T_s}\right) = \frac{1}{T_s}\sum_k \delta\left(f-\frac{k}{T_s}\right) \quad \square$$

A different look:

Consider the DT signal $x[n] = e^{i2\pi f_0 n}$ (If $f_0 = 0$, this is $x[n]=1$)

CT representation

$$x(t) = \sum_k x[k]\,\delta(t-k)$$

CT representation

$x(t) = \sum_k x[k]\,\delta(t-k)$

$= \sum_k \delta(t-k)$

What is $X(f)$?

In CT: $e^{j2\pi f_0 t} \xrightarrow{\mathcal{F}} \delta(f-f_0)$

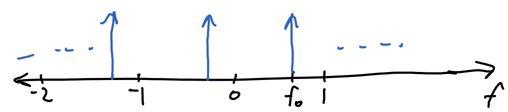
We verify with the inverse transform integral.

Same technique works here:

i.e. $$\int_{[\text{one period}]} \delta(f-f_0)\,e^{j2\pi f n}\,df = e^{j2\pi f_0 n}$$

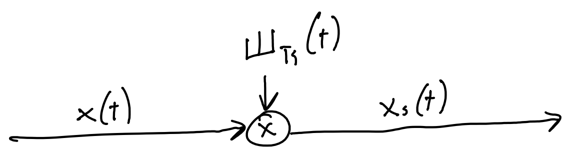
However, $X(f)$ has period 1 for DT $x[n]$

$$\Rightarrow X(f) = \sum_{k=-\infty}^{\infty} \delta(f - f_0 - k)$$

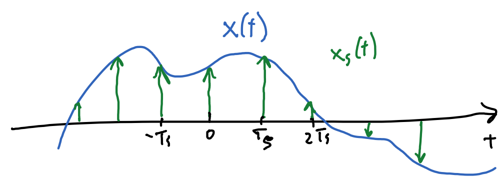


Sampling:

$x[n] = x(nT_s) \quad \forall n$



i.e. $x_s(t) = x(t)\,Ш_{T_s}(t)$

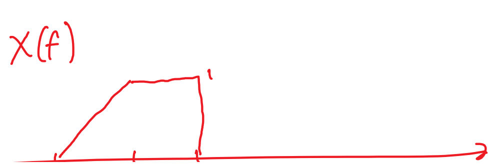


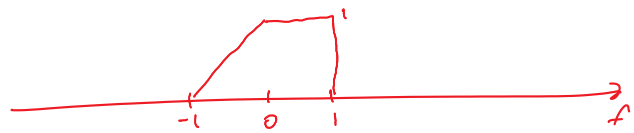
$x[n]$ is the DT equivalant of $x_s(t)$

Use multiplication rule:

$$X_s(f) = X(f) * \frac{1}{T_s} Ш_{\frac{1}{T_s}}(f)$$

Exercise:





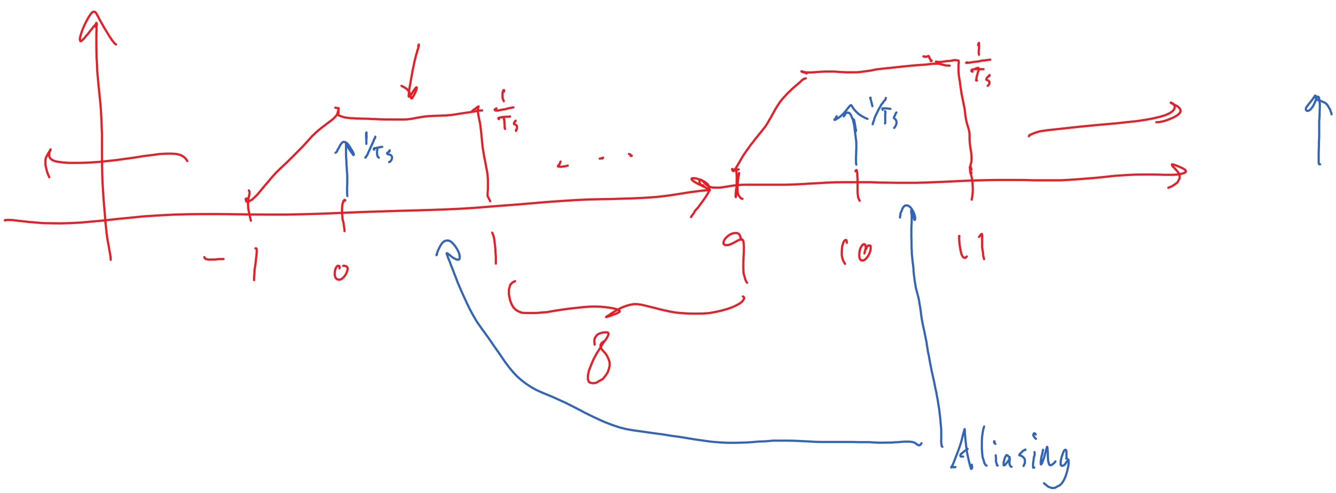
Suppose $T_s = \frac{1}{10}$

What is $X_s(f)$?

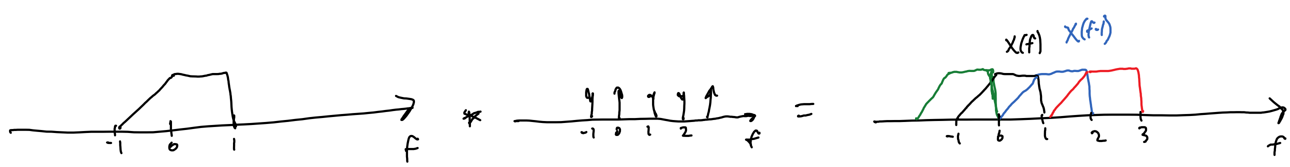
$$X(f) * Ш = \int_{-\infty}^{\infty} X(\tau)\, Ш(f-\tau)\, d\tau =$$

$$X(f) * \frac{1}{T_s}\sum_k \delta\left(f - k\frac{1}{T_s}\right) =$$

$$\sum_k X(f) * \delta(f - kT_s) = \sum_k X\left(f - k\overset{10}{\cancel{T_s}}\right)$$



What if $T_s = 1$?



Examples of aliasing artifact:
- See lecture 3 audio example
- wheels in video https://www.youtube.com/watch?v=jHS9JGkEOmA
- spacial aliasing in photo http://paulbourke.net/miscellaneous/aliasing/aliasing3.gif

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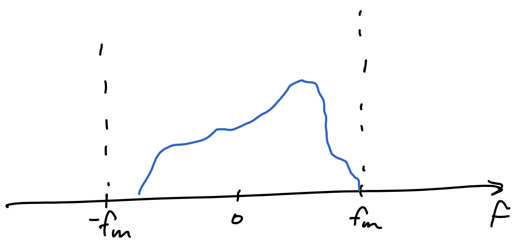
Band limited: $X(f) = 0 \quad \forall |f| > f_m$

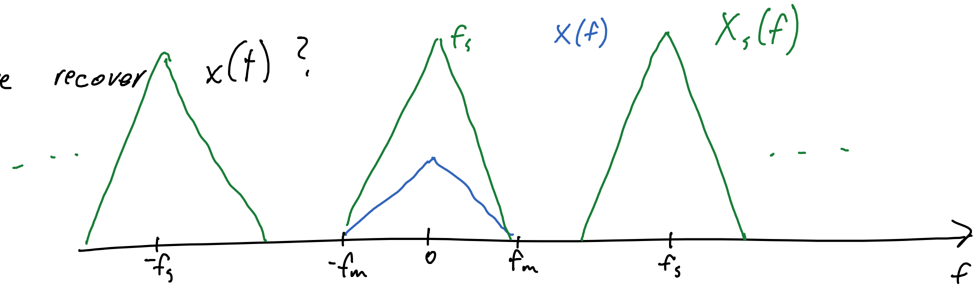
f_m ← maximum frequency



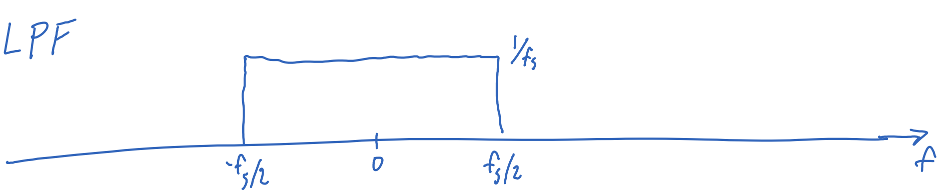
Sampling Theorem: (Nyquist) If $f_s = \frac{1}{T_s} > 2 f_m$ then $x(t)$ is uniquely determined by the samples $x_s(t)$.

Proof: See exercise above. No overlap caused by aliasing.

How do we recover $x(t)$?

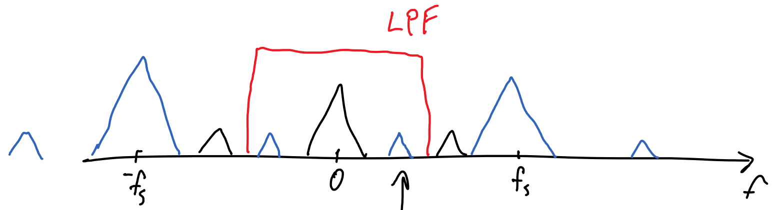


Ideal LPF



In time domain: $* \operatorname{sinc}(f_s t)$ i.e. sinc interpolation!

What happens if the sample rate is too low?



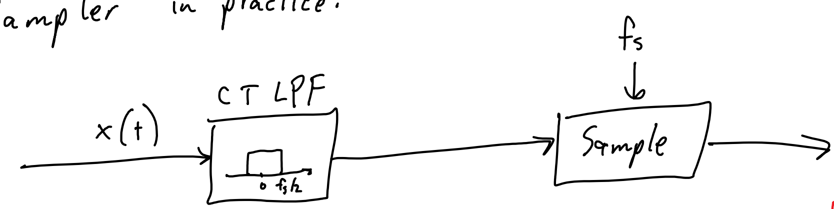
Aliasing artifact. New frequencies are introduced.

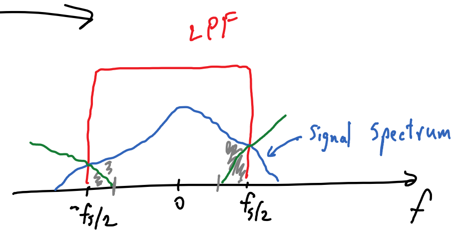
Practical Considerations:

Anti-aliasing filter:

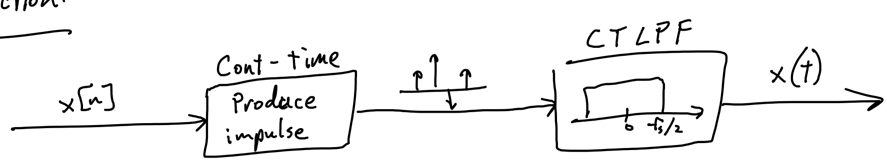
LPF to avoid aliasing $\boxed{\text{before sampling}}$

Sampler in practice:





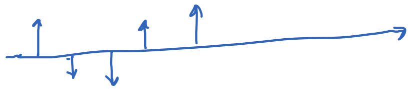
Reconstruction:



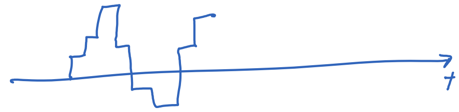
Two constraints to consider:

1.) How do you do this?

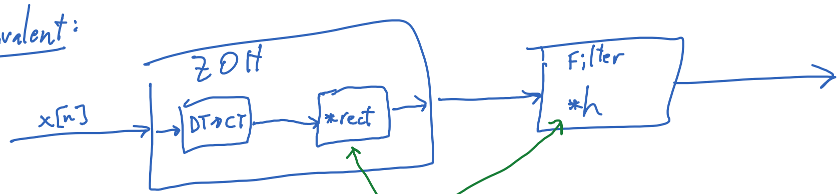
ii) How do you do ?



Easy to produce zero-order hold:

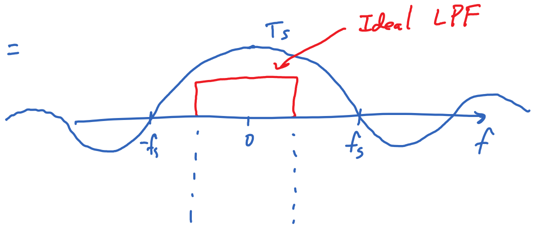


Equivalent:

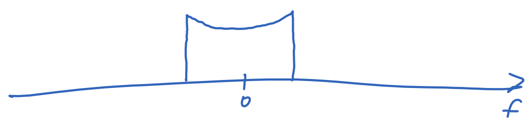


Want combination to be ideal LPF.

$* \text{rect}(t/T_s)$ =



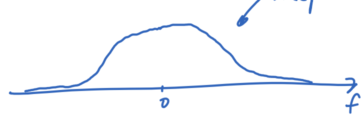
Let $H(f)$



2.) Can't implement ideal LPF:

- non-causal
- output depends on samples arbitrarily far away in time.
- circuit.

steepness depends on circuit.



Vocabulary:

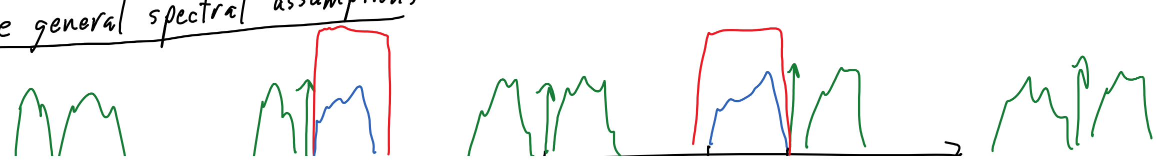
Bandwidth $= f_m$

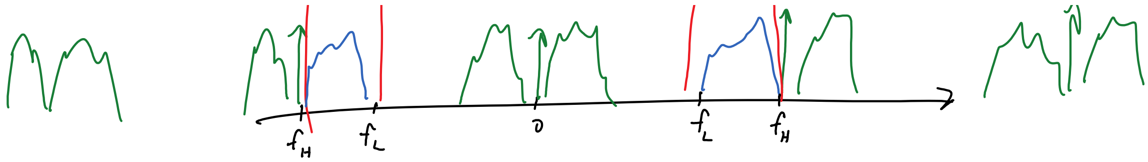
Nyquist rate $= 2f_m$

(if there is upper and lower boundary it is the difference.

Some time people call the bandwidth $2f_m$ or the Nyquist "frequency" f_m.

More general spectral assumptions:





Bandwidth = $f_H - f_L$

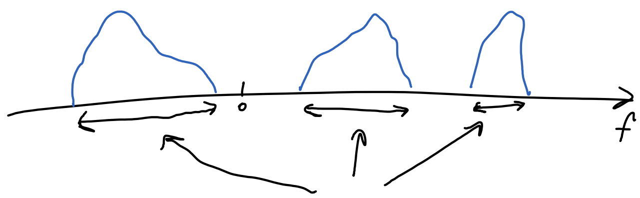
Cannot naively apply sampling theorem at twice bandwidth $f_H - f_L$.

Always okay to use $f_s > 2f_H$, but lower f_s might work also, with different reconstruction method.

Example: If $f_L > \frac{1}{2} f_H$

Sample at $f_s = f_H$

Suppose the spectrum is contained in some set of intervals:



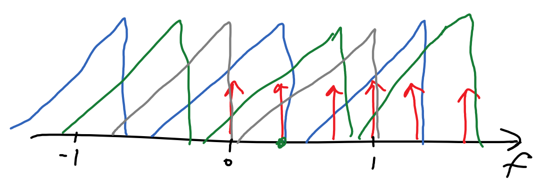
Landau rate = sum of interval widths

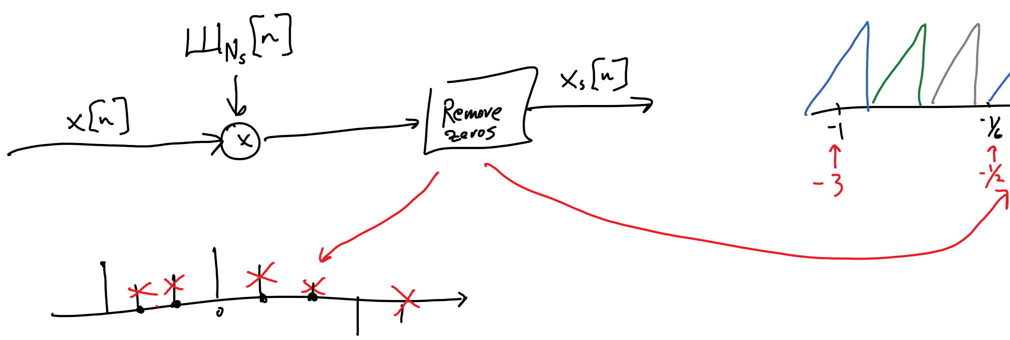
Landau rate is lower bound on minimum sample rate.

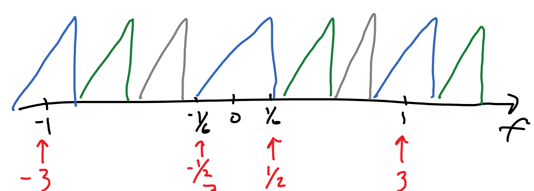
DT Sampling:

"Down-Sampling"

Just like CT sampling.







Reconstruction.
"Up-sampling"

Add zero's then LPF (in discrete-time).