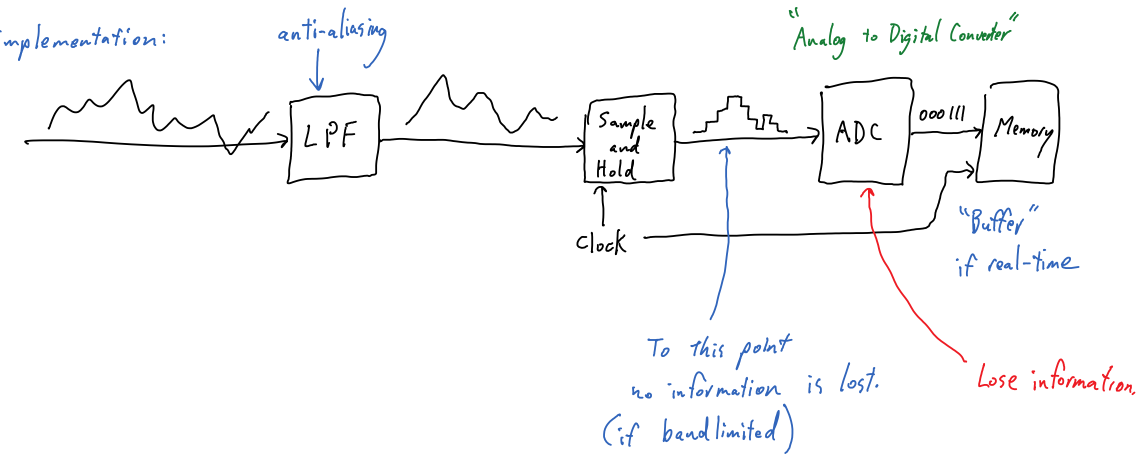
Lecture 18

Saturday, March 28, 2015 9:07 PM

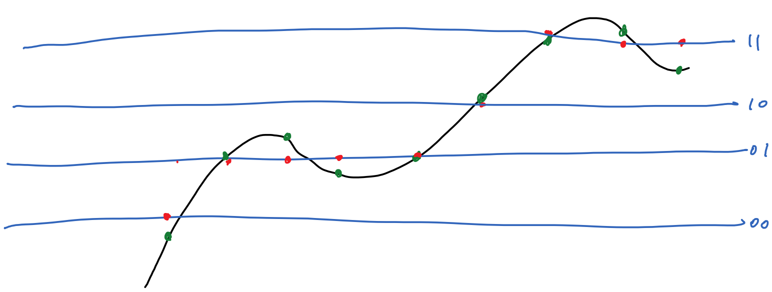
Analog | **Digital**

- CT
  Domain and range are $\mathbb{R}$ $\xrightarrow{\text{Sampling and Quantization}}$ Finite set of values

Implementation:



Example: 2-bit quantization



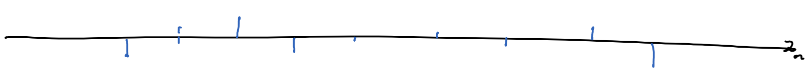
00, 01, 01, 01, 10, 11, 11, 11

Let $\hat{x}[n]$ be the reconstructed signal (red).

Then $\text{error}[n] = x[n] - \hat{x}[n]$

error[n]

"Noise"



CD Audio: 44.1 kHz Sample rate
2 Channels (stereo)
16 bit quantization (or 24)

$2^{16} = 65{,}536$ quant. points.

$\Rightarrow 44{,}100 \times 2 \times 16 = 1{,}411{,}200$ bits/sec.
176,400 bytes/sec.
10,584,000 bytes/minute.

10 MB/min

Doesn't count overhead.
Not even highest quality.

/min

Not even highest quality.

Probability Primer:

X is a random object with outcomes in the set Ω.

e.g. Represents whether a team wins or loses a game.

$$\Omega = \{\text{win}, \text{loss}, \text{tie}\}$$

$P(X=\text{win}) = 70\%$
$P(X=\text{loss}) = 20\%$
$P(X=\text{tie}) = 10\%$

Non-negative. Sum to 1.

$$P(\overbrace{X=\text{win} \text{ or } X=\text{loss}}^{\text{event (set of outcomes)}}) = 70\% + 20\% = 90\%$$

Random Variable: $\Omega \subset \mathbb{R}$

Discrete: X takes one of finitely many (or countably-many) values.

Probability mass function:

$$p_X(a) = \text{Prob}(X=a) \quad \forall a.$$

$$\Rightarrow \text{Prob}(X \in S) = \sum_{a \in S} p_X(a)$$

Rules: $p_X(x) \geq 0 \quad \forall x$

$\sum_x p_X(x) = 1$

Example:

$p_X(x)$

$$p_X(x) = \begin{cases} \frac{1}{2}, & x=0 \\ \frac{1}{4}, & x=1,2 \\ 0, & \text{else} \end{cases}$$

$EX = \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2$
$= 3/4$

$\text{Var}(X) = \frac{1}{2}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{5}{4}\right)^2$
$= \frac{44}{64}$

Continuous: Probability Density Function:

$$f_X(a) \qquad \text{For } S \subset \mathbb{R}, \quad P(X \in S) = \int_S f_X(a)\,da$$

Rules: $f_X(x) \geq 0 \quad \forall x$

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Examples:

Uniform Distribution:

$$\text{Unif}[b,c] = f_X(x) = \begin{cases} \frac{1}{c-b}, & b \leq x \leq c \\ 0, & \text{else} \end{cases}$$

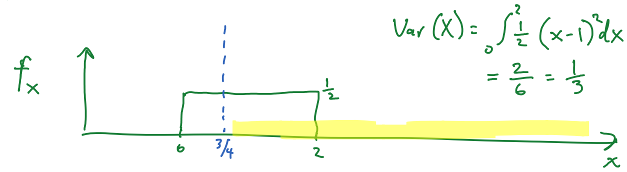
in general

$EX = \frac{b+c}{2}$

$\text{Var}(X) = \frac{(c-b)^2}{12}$

$$EX = \int_0^2 \frac{1}{2}\cdot x\,dx = \frac{1}{4}x^2 \Big|_0^2$$

Consider $\text{Unif}[0,2]$

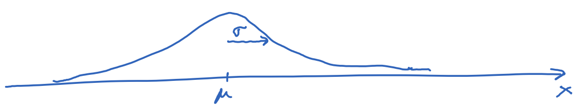


$$EX = \int_0^2 \frac{1}{2} \cdot x \, dx = \frac{1}{4} x^2 \Big|_0^2 = 1$$

$$\text{Var}(X) = \int_0^2 \frac{1}{2} (x-1)^2 dx = \frac{2}{6} = \frac{1}{3}$$

$$P(X \geq 3/4) = 5/8$$

Gaussian : $\mathcal{N}(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



$EX = \mu$
$\text{Var}(X) = \sigma^2$
Standard deviation $= \sigma$

Expected Value:

Mean of $g(X)$ $\quad E\, g(X) = \sum_x p_X(x) g(x) \longleftarrow$ Dicrete

$$E\, g(X) = \int_{-\infty}^{\infty} f_X(x) g(x) dx \longleftarrow \text{Continuous}$$

Mean of X. First moment. $\quad EX = \sum_x p_X(x) \cdot x$

Law of Large Numbers: Let $X_1, X_2, \ldots, X_n$ be independent R.V.'s each have dist. p_X.

Sample Ave. $\quad \frac{1}{n} \sum_{i=1}^{n} g(X_i) \longrightarrow E\, g(X)$

Second Moment: EX^2

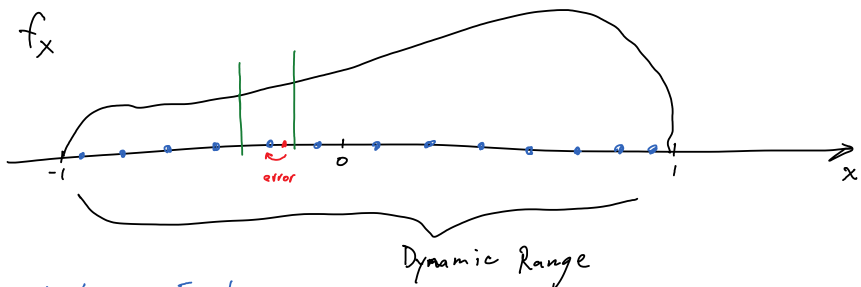
LLN : $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \longrightarrow EX^2$ (Power)

Variance: $E(X - EX)^2$

Standard Deviation: $\sqrt{Var(X)}$

Quantization:

Assume a distribution of signal $x[n]$.



Uniform Quantization: Equal spacing.

Error is uniformly distributed if quantization is fine enough.

$$|error| \le \frac{DR}{2\,(\#\text{ Quant. points})}$$

$$= \frac{DR}{2 \cdot 2^b}$$

Power of Quant. Noise (error): $\dfrac{\left(\frac{DR}{2^b}\right)^2}{12}$