Circular Convolution and Phase

Convolution for periodic signals:

Let $x(t)$ and $y(t)$ have period T.

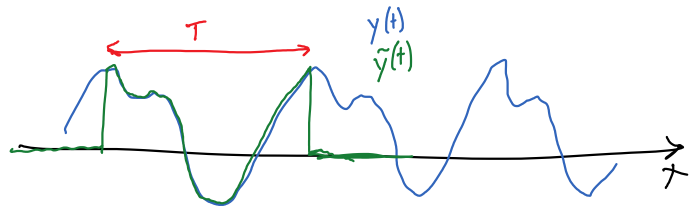
$$x(t) * y(t) = \int_{-\infty}^{\infty} \underbrace{x(\tau)\, y(t-\tau)}_{\text{periodic}}\, d\tau = \infty$$

$$x(t) \circledast_T y(t) = \int_{\langle T \rangle} x(\tau)\, y(t-\tau)\, d\tau$$

or equivalently

Let $\tilde{y}(t)$ be one period $y(t)$

$$x(t) * \tilde{y}(t) = x(t) \circledast_T y(t)$$



Fourier Properties:

DT: Multiplication rule $\quad x[n] \cdot y[n] \xrightarrow{\mathcal{F}} X(f) \circledast_1 Y(f)$

CTFS: Circular Convolution rule: $\quad$ Period T

$$x(t) \longrightarrow \{a_k\}$$
$$y(t) \longrightarrow \{b_k\}$$
$$x(t) \circledast_T y(t) \longrightarrow \frac{1}{T} a_k b_k$$

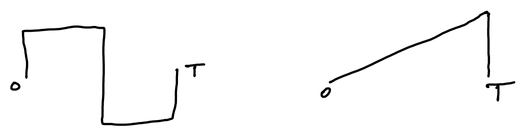
Two finite-duration signals:

Duration T for both.

Regular convolution

Assume zero elsewhere.
(Matlab would do this)

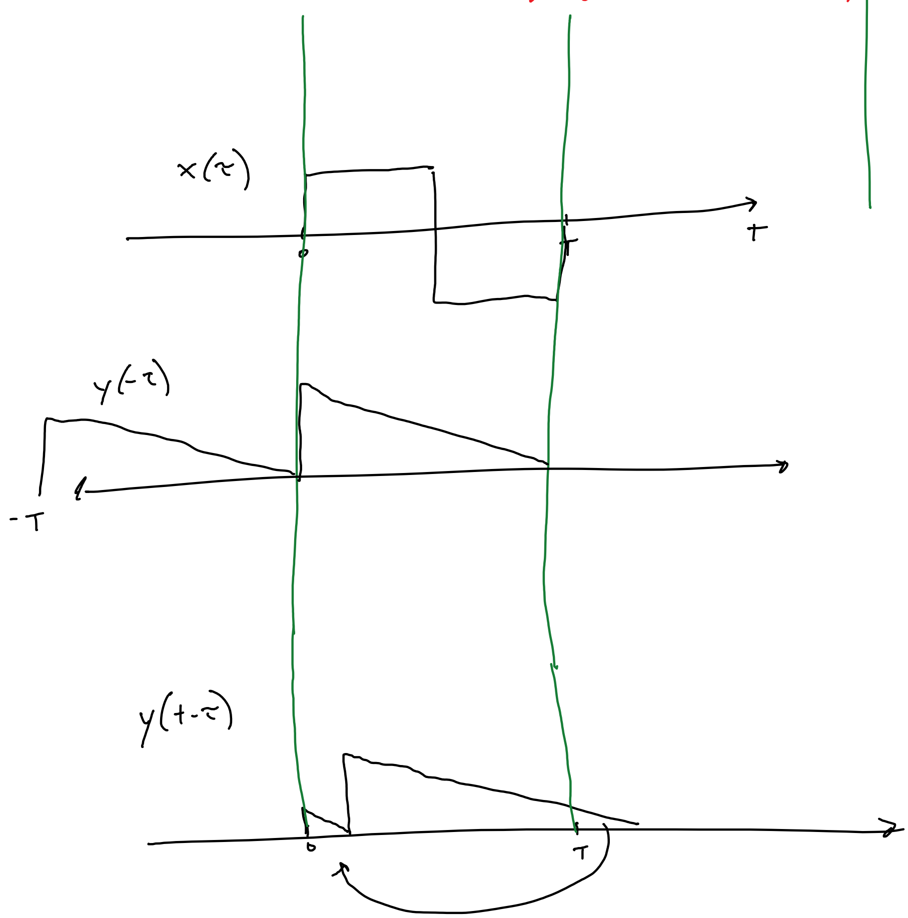
Duration T for both.



Circular conv. results in same length T.

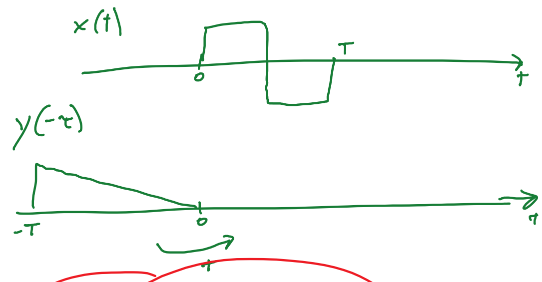
$$x(t) \circledast_T y(t) = \int_0^T x(\tau)\, y(t-\tau \bmod T)\, d\tau$$

Add multiples of T such that $t-\tau+kT \in (0,T)$

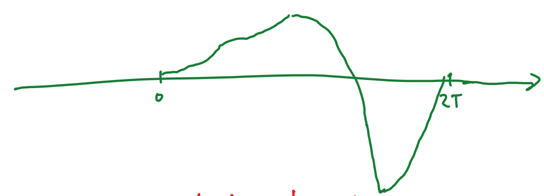


Assume zero elsewhere. (Matlab would do this)

Flip-and-shift



Overlap for $t \in (0, 2T)$



DT: $2N-1$ length.

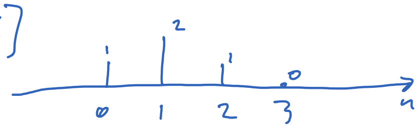
Equivalently: Circular convolution of periodic extensions.

(See Matlab demo: Circular convolution through DFT multiplication.)

Example of circ. conv.:

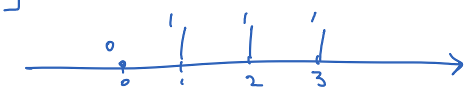
Duration $N=4$

$x[K]$



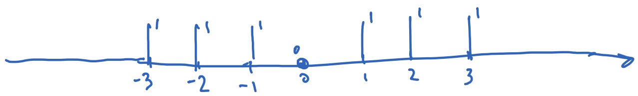
$x = (1, 2, 1, 0)$

$y[K]$



$y = (0, 1, 1, 1)$

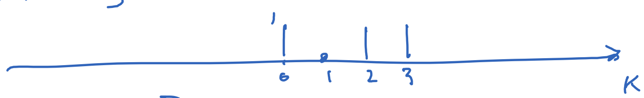
$y[-K]$



$z[n] = x[n] \circledast_4 y[n]$

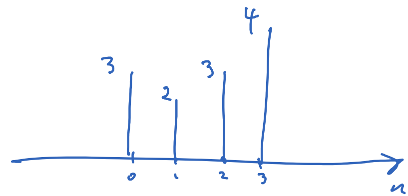
$z[0] = 3$

$y[1-K \bmod 4]$

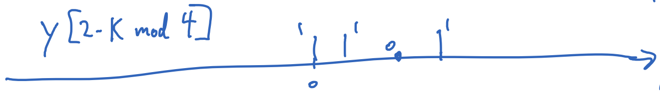


$z[1] = 2$

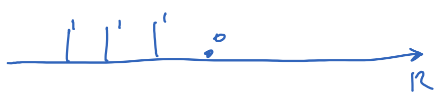
$z[n]$



$y[2-K \bmod 4]$



$z[2] = 3$



$z[3] = 4$

Why Circ. Conv. for DT Multiplication Property?

$x[n] \cdot y[n]$

c.t.; interpolation

$$\left(\sum_n x[n]\,\delta(t-n)\right) \cdot \left(y(t) \right) = \sum_n x[n]\cdot y[n]\,\delta(t-n)$$

one period of $Y(f)$

Freq. Domain:

$$X(f) * \tilde{Y}(f) = X(f) \circledast_1 Y(f)$$

How does phase affect a signal?

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See Matlab demo. Timing is in the phase.