Lecture 26

Wednesday, April 15, 2015 11:55 AM

Laplace Transform (CT only)

Why? - Signals that don't have F.T. (infinite energy)
- Unstable systems (diff. eq.)

$$\underset{\text{complex}}{X(s)} = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\text{signal}} e^{-st}\, dt$$

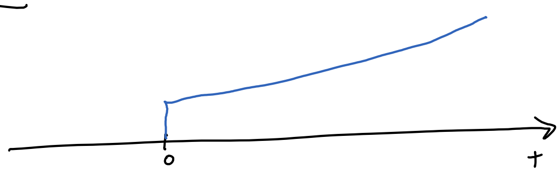
The domain of $X(s)$ is complex plane.

May only converge for some s.

Two parts: 1.) Region of Convergence (ROC)
2.) $X(s)$

Examples:

$x(t) = e^{t}u(t)$



Notice: No Fourier Transform
Energy is infinite.

$$X(s) = \int_{-\infty}^{\infty} e^{t}u(t)\, e^{-st}\, dt$$

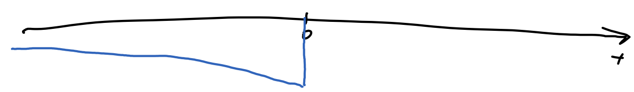
$$= \int_{0}^{\infty} e^{(1-s)t}\, dt$$

$$= \frac{1}{1-s} e^{(1-s)t}\Big|_{t=0}^{\infty}$$

$$= \frac{-1}{1-s} = \boxed{\frac{1}{s-1}}$$

ROC: $\boxed{Re\{s\} > 1}$

$x(t) = -e^{t}u(-t)$



$$X(s) = -\int_{-\infty}^{\infty} e^{t}u(-t)\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{(1-s)t}\, dt$$

$$= \frac{-1}{1-s} e^{(1-s)t}\Big|_{t=-\infty}^{0}$$

$$= \boxed{\frac{1}{s-1}}$$

ROC: $Re\{1-s\} > 0 \Rightarrow \boxed{R\{s\} < 1}$

$$e^{(1-s)t} = e^{(1-\mathrm{Re}\{s\})t}\, e^{-i\,\mathrm{Im}\{s\}t}$$

Exp growth — periodic

Relationship to F.T.:

Let $s = \sigma + i2\pi f$ (real: σ, f)

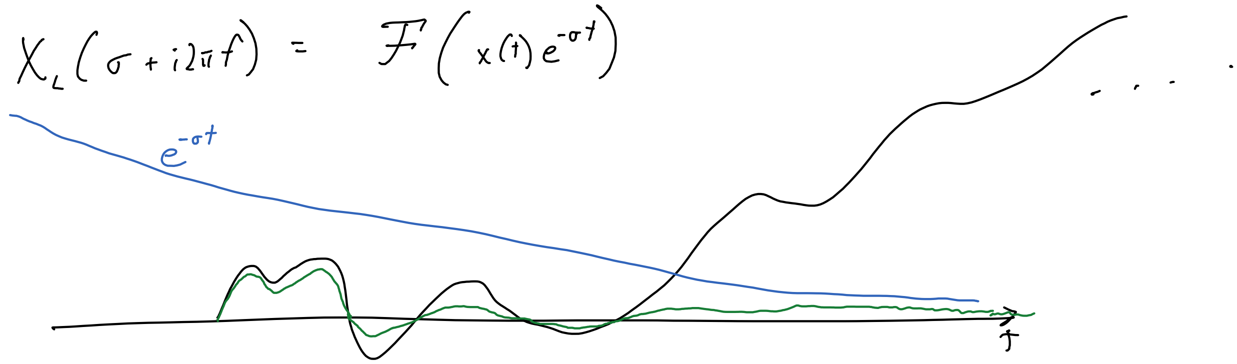
$$X_L(\sigma + i2\pi f) = X(s)\Big|_{s=\sigma+i2\pi f} = \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t} e^{-i2\pi f t}\, dt = \mathcal{F}\left(x(t)e^{-\sigma t}\right)$$

$$X_L(i2\pi f) = X(s)\Big|_{s=i2\pi f} = \mathcal{F}\left(x(t)\right)$$

$$X_L(i2\pi f) = X_F(f)$$

Interpret Laplace Transform:

$$X_L(\sigma + i2\pi f) = \mathcal{F}\left(x(t)e^{-\sigma t}\right)$$

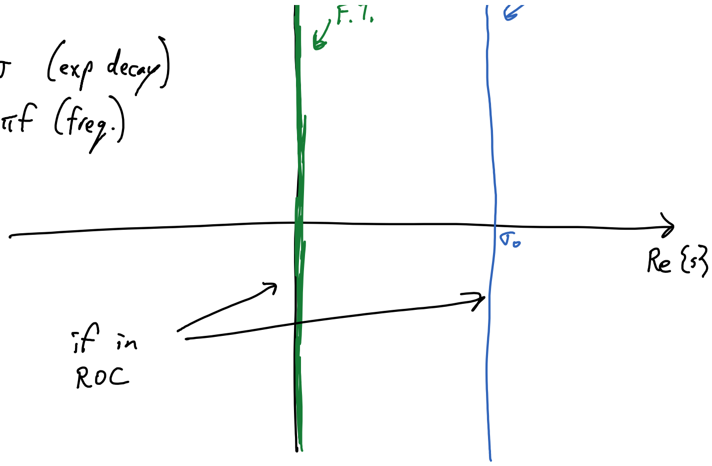


- If σ is large enough, F.T. might converge (finite energy).
- If σ is too large or too small, can cause divergence. (consider left side of signal)

$\mathrm{Re}\{s\} = \sigma$ (exp decay)

$Re\{s\} = \sigma$ (exp decay)

$Im\{s\} = 2\pi f$ (freq.)



Laplace transform is redundant.

Example: $x(t) = rect(t)$

$$X(s) = \int_{-\infty}^{\infty} rect(t)\, e^{-st}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-st}\, dt$$

$$= \frac{e^{-s/2} - e^{s/2}}{s}$$

ROC: All s. ROC = complex plane. $\Leftarrow$ Finite-duration

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Inverse of Laplace Transform:

1.) If imag. axis is in the ROC, take the inverse F.T. of $X_L(i2\pi f)$.

2.) In general:

$$x(t) = \lim_{f \to \infty} \frac{1}{i2\pi} \int_{\sigma - i2\pi f}^{\sigma + i2\pi f} X(s)\, e^{st}\, ds$$

(for any σ that puts this line in the ROC.

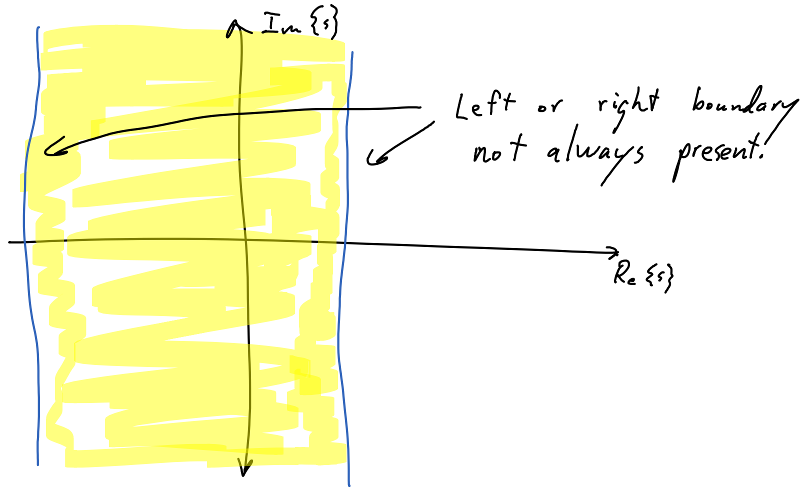
$\sigma - i2\pi f$

for any σ that puts this line in the ROC.

$$e^{\sigma t} \int_{-\infty}^{\infty} X_L(\sigma + i2\pi f)\, e^{i2\pi f t}\, df.$$

ROC: Always vertical strips

Example



Stability: Let $h(t)$ be the impulse response of a system.

The system is stable if ROC of $H(s)$ contains the imag. axis in the interior

Not on boundary.

Example of unusual case:

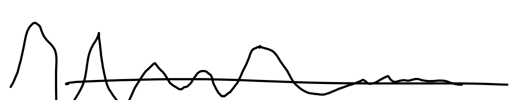
$h(t) = \text{sinc}(t)$

$X(s) = \int_{-\infty}^{\infty} \text{sinc}(t)\, e^{-st}\, dt$

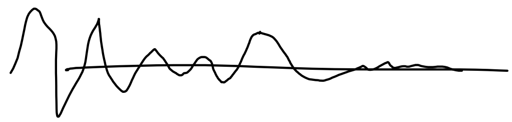
Laplace transform:

$\boxed{\text{ROC: } \operatorname{Re}\{s\} = 0.}$

$e^{\sigma t}\, \text{sinc}(t)$ for $\sigma \neq 0$

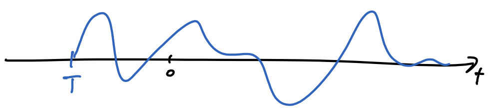


ROC: Re(s) = 0.



Characterization of signal:

1.) Right-sided signal



$\exists\, T$ s.t. $x(t) = 0 \quad \forall t < T$

$\Rightarrow$ ROC has no right boundary

2.) Left-sided signal

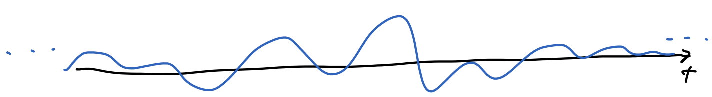
$\exists\, T$ s.t. $x(t) = 0 \quad \forall t > T$

$\Rightarrow$ ROC has no left boundary

3.) Two-sided

(e.g. sinc(t))

(neither left or right sided.) $\leftarrow$ unfortunate naming



4.) Finite-duration

(Mathematically: both left and right sided.)

$\Rightarrow$ ROC has no left or right boundary (ROC = complex plane)

Causal:

$h(t) = 0, \quad \forall t < 0$, $\leftarrow$ "causal" signal

Impulse response

$\Rightarrow$ right-sided impulse response.

Rational Laplace Transform:

Rational function : $\dfrac{N(s)}{D(s)}$ ← Polynomial

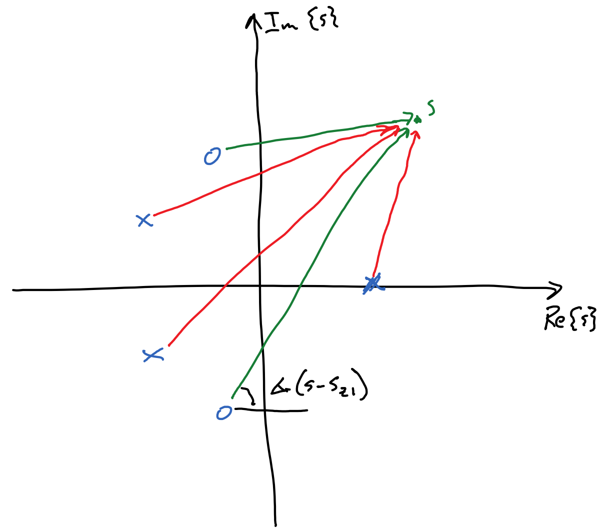
Factored : $C \dfrac{(s-s_{z_1})(s-s_{z_2})\cdots}{(s-s_{p_1})(s-s_{p_2})\cdots}$ ← # of terms is order of polynomial.

s_{z_i} are the zeros.

s_{p_i} are the poles.

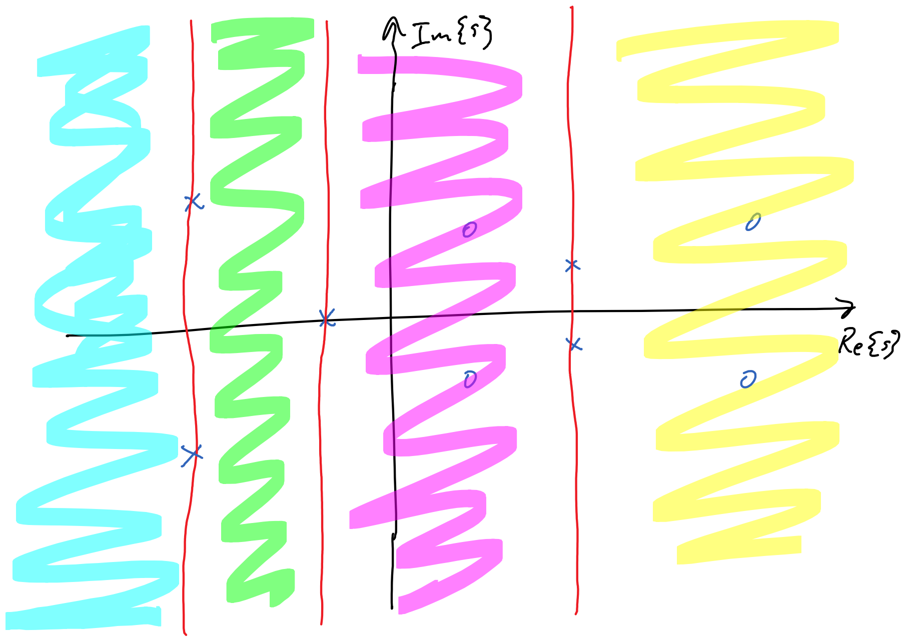
Magnitude : $|C| \dfrac{|s-s_{z_1}|\,|s-s_{z_2}|\cdots}{|s-s_{p_1}|\,|s-s_{p_2}|\cdots}$

Phase : $\measuredangle C + \sum_{\text{zeros}} \measuredangle (s-s_{z_i}) - \sum_{\text{poles}} \measuredangle (s-s_{p_i})$



For Rational Laplace Transforms:

Boundaries of ROC are the poles.



Quickly check whether a rational Laplace transform could be both causal and stable.

All poles are in the left half-plane.
(real parts negative)