Lecture 27

Monday, April 20, 2015 9:29 PM

Z - Transform

D.T. signal:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

z is complex
May converge for some z.

Same as Laplace transform: $z = e^s$

For Laplace, $\mathrm{Re}\{s\}$ determined convergence.

i.e. $s = \underline{\sigma} + i2\pi f$

$z = e^s = e^{\sigma} e^{i2\pi f}$

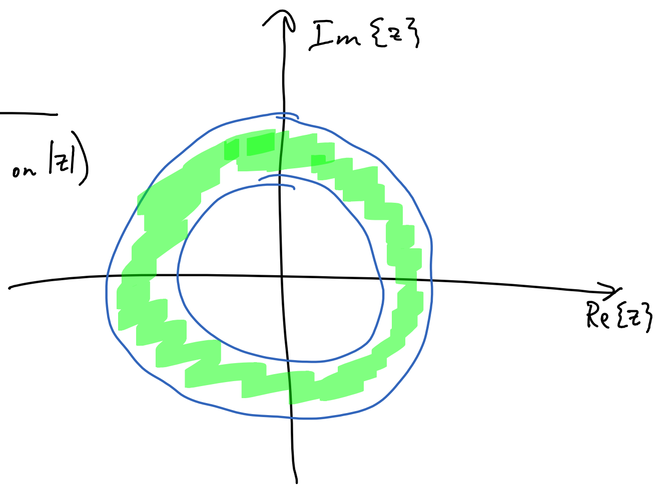
$|z| = e^{\sigma}$

Let $z = a e^{i2\pi f}$

$$X_z(a e^{i2\pi f}) = \sum_{n=-\infty}^{\infty} x[n] a^{-n} e^{-i2\pi f n} = \mathcal{F}(x[n] a^{-n})$$

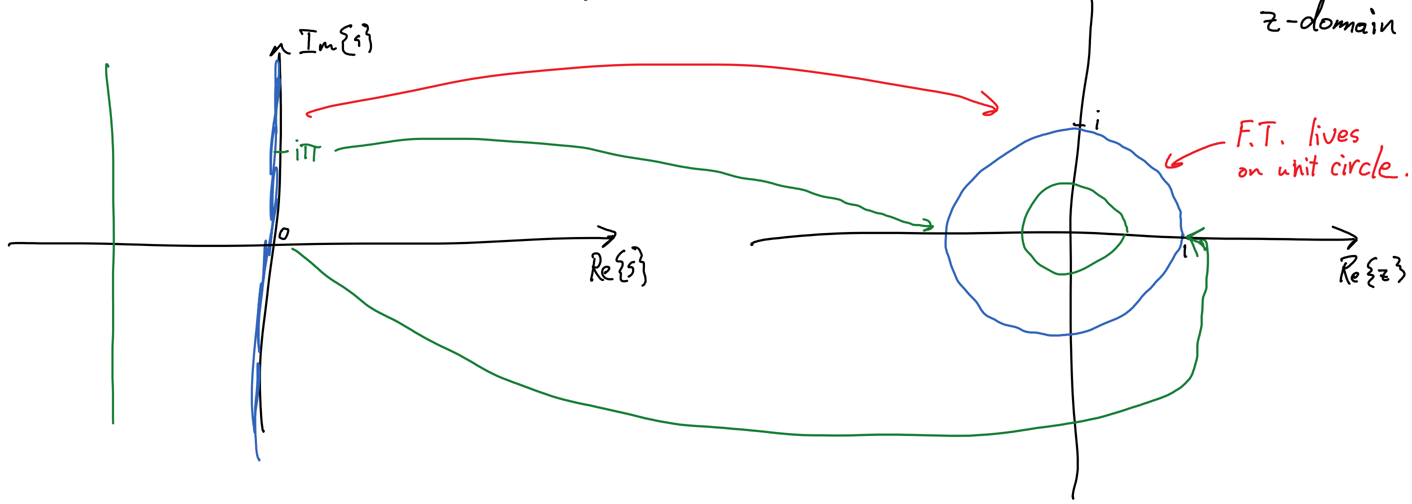
$$X_z(e^{i2\pi f}) = X_F(f)$$

ROC is a ring:
(i.e. convergence depends on $|z|$)



The s-plane has been wrapped in circles by $z = e^s$.

Substitution $z = e^s$



ROC : Rings defined by $|z| \in (a, b)$

Right-sided: ROC has no outward boundary.

Left-sided: ROC has no inner boundary.
(may have pole at $z = 0$.)

Finite-duration: ROC is the whole space (may have pole at $z=0$)

Stability: $\Rightarrow$ Unit circle is in the ROC.
$\Leftarrow$ Unit circle is in interior of the ROC

Why substitute $z = e^s$ for D.T.?

1.) Emphasizes periodicity
2.) Rational z-transforms

F.T. Notation from book: C.T.F.T. $X(i2\pi f)$ ($i \to j$, $2\pi f \to \omega$)

D.T.F.T. $X(e^{i2\pi f})$

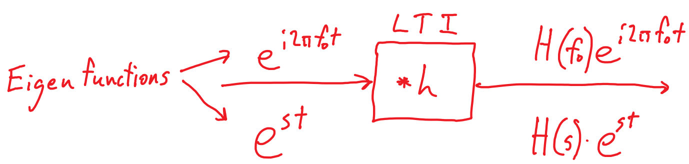
This notation comes from substituting into the Laplace and z transform.

Properties of Laplace and Z-transforms:

Linearity: $a\,x(t) + b\,y(t) \xrightarrow{\mathcal{Z}} a\,X(s) + b\,Y(s)$

Convolution: $x(t) * y(t) \xrightarrow{\mathcal{Z}} X(s)\cdot Y(s)$

ROC is intersection.



Derivative Property: $\frac{d}{dt}x(t) \xrightarrow{\mathcal{L}} s\,X(s)$ No change to ROC.

Integral Property: $\int_{-\infty}^{t} x(\tau)\,d\tau \xrightarrow{\mathcal{Z}} \frac{1}{s}X(s)$ ROC: $Re\{s\} > 0$ intersect with ROC of $X(s)$.

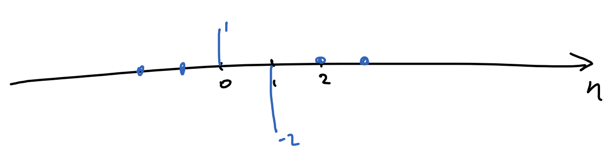
not rational

C.T. Time-shift Property: $x(t-T) \xrightarrow{\mathcal{Z}} e^{-Ts}X(s)$

D.T. Time-shift Property: $x[n-N] \xrightarrow{\mathcal{Z}} z^{-N}X(z)$

rational

Example: $x[n] = \delta[n] - 2\delta[n-1]$



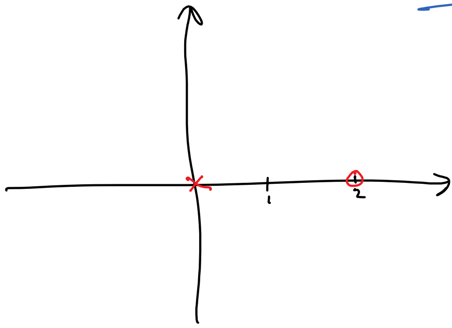
$$X(z) = \sum_{n=-\infty}^{\infty} \left(\delta[n] - 2\delta[n-1]\right) z^{-n}$$

$$= 1 - 2z^{-1}$$

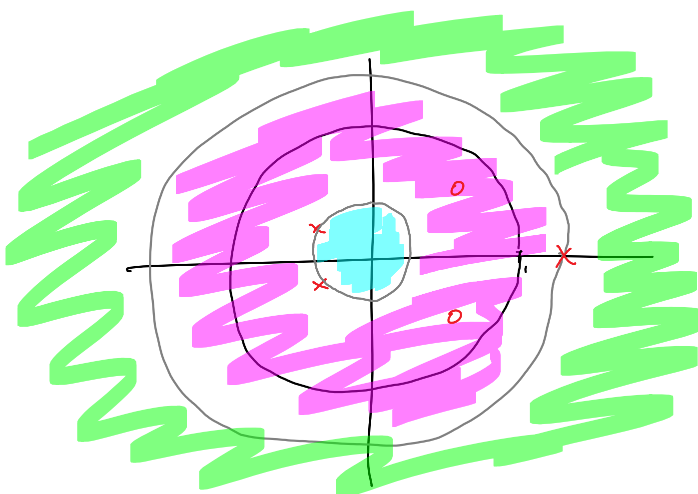
Is this rational?

Yes.

$$\frac{z-2}{z}$$



For rational z-transform: Poles are at the boundaries of ROC.



left-sided. Not stable.

Two-sided stable.

right-sided not stable. causal.

What about causality?

For rational z-transform

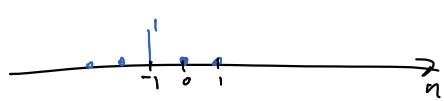
$$\boxed{\text{Right-sided} \Rightarrow \text{causal} \quad \text{if and only if} \quad \#\text{zeros} \le \#\text{poles}.}$$

(Note: For rational Laplace transforms, right-sided $\Leftrightarrow$ causal, unconditionally.)

Example:

$$x[n] = \delta[n+1]$$

non-causal



right-side

$$X(z) = z$$

\# zeros = 1

\# poles = 0

Justification:

Assume $x[n]$ is non-causal

For some positive K.

$$\sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[-K] z^{K} + \cdots$$

order of numerator > order of denominator