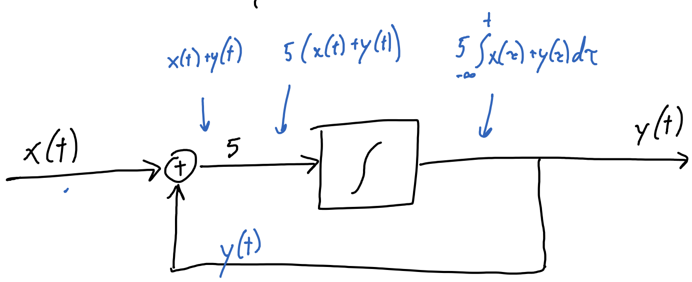
Lecture 28

Wednesday, April 22, 2015 3:18 PM

Stability of Causal Systems



$$y(t) = 5\int_{-\infty}^{t} \left(x(\tau)+y(\tau)\right)d\tau$$

$$y'(t) = 5\left(x(t)+y(t)\right)$$

$$y(t) = -x(t) + \frac{1}{5}y'(t)$$

LTI System: Impulse response $h(t)$ fully describes the system.
Transfer function $H(s)$.

Constant Coefficient Linear Differential (Difference) Equation:

(LTI)

Differential Equation:

order

$$\sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) = \sum_{k=0}^{M} a_k \frac{d^k}{dt^k} y(t)$$

$$\sum_{k=0}^{M} b_k s^k X(s) = \sum_{k=0}^{M} a_k s^k Y(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)}} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{M} a_k s^k}$$

Transfer function

Rational Function

Difference Equation:

e.g. $y[n] = 3x[n] + 2x[n-1] - \frac{1}{2}y[n-1]$

$$\sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} a_k y[n-k]$$

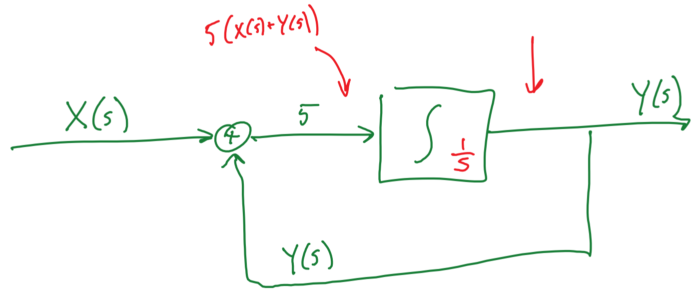
$$\sum_{k=0}^{M} b_k z^{-k} X(z) = \sum_{k=0}^{M} a_k z^{-k} Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{M} a_k z^{-k}}$$

Rational

Same example:

Same example:



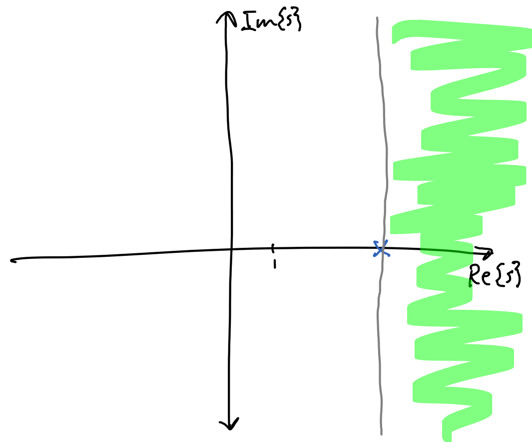
$$Y(s) = \frac{5}{s}\left(X(s) + Y(s)\right)$$

$$(s-5)Y(s) = 5\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \boxed{\frac{5}{s-5}}$$

No zeros
1 pole

Assume causal:



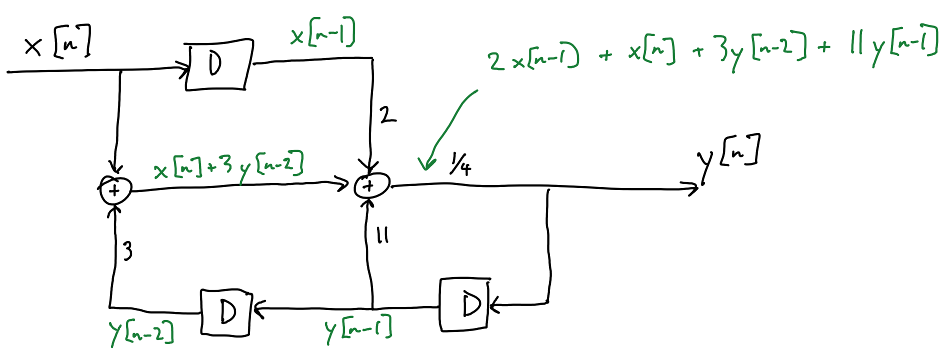
$$h(t) = 5e^{5t}u(t)$$

Stable if and only if:
(causal and rational)

CT → All poles in left half-plane (Real part negative)

DT → All poles in the unit circle.

DT Example:



$$y[n] = \frac{1}{4}\left(2\,x[n-1] + x[n] + 3y[n-2] + 11y[n-1]\right)$$

Impulse response: $\quad 0 \quad 1 \quad 0 \quad 0$

$n=0 \quad y[0] = \frac{1}{4}$

$n=1 \quad 1 \quad 0 \quad 0 \quad \frac{1}{4}$

$y[1] = \frac{1}{2} + \frac{11}{16}$

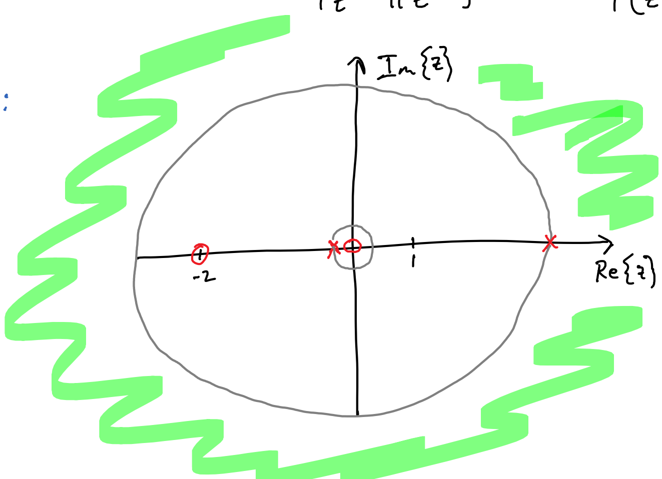
$$4y[n] - 11y[n-1] - 3y[n-2] = x[n] + 2x[n-1]$$

Use z-transform:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{4 - 11z^{-1} - 3z^{-2}}$$

$$= \frac{z^2+2z}{4z^2-11z-3} = \frac{z(z+2)}{4(z-3)(z+\frac{1}{4})}$$

Assume causal:



Long-division:

$$H(z) = \frac{z^2+2z}{4z^2-11z-3} = \frac{1}{4} + \frac{19}{16}z^{-1} + \frac{\frac{11\cdot 19}{16} + \frac{3\cdot 19}{16}z^{-1}}{4z^2-11z-3}$$

$$\begin{array}{r l}
 & \frac{1}{4} + \frac{19}{16}z^{-1} \\
4z^2-11z-3 \,\big) & \cancel{z^2} + 2z \\
 & -(\cancel{z^2} - \frac{11}{4}z - \frac{3}{4}) \\
\hline
 & \frac{19}{\cancel{4}}z + \frac{3}{4}
\end{array}$$

$$\frac{1}{4}\delta[n] + \frac{19}{16}\delta[n-1] +$$

$$\frac{\cancel{19}}{\cancel{4}} z + \frac{3}{4}$$

$$-\left(\frac{\cancel{19}}{\cancel{4}} z - \frac{11\cdot 19}{16} - \frac{3\cdot 19}{16} z^{-1}\right)$$

Inverse Laplace Transform and Z-transform:

Important Transform Pairs:

Laplace:

$e^{-at}u(t)$	$\frac{1}{s+a}$	$Re\{s\} > Re\{-a\}$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$Re\{s\} < Re\{-a\}$
$\boxed{t^K e^{-at}u(t) \quad \frac{K!}{(s+a)^{K+1}}}$		$Re\{s\} > Re\{-a\}$

Z-transform

$\delta[n]$	1	
$\delta[n-K]$	z^{-K}	
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$n a^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$

Only relevant pairs for causal systems with rational transfer functions.
(hi her order poles in the Z-transform have a complicated expression.)

Method: Partial Fraction Expansion:

$$\frac{z+2}{4(z-3)(z+\frac{1}{4})} = \frac{A}{z-3} + \frac{B}{z+\frac{1}{4}}$$