Lecture 3

Tuesday, February 10, 2015

8:00 PM

Sinusiods : $\cos(2\pi f t + \phi)$ ← these

Surprise! Any real periodic function can be decomposed into these

$$\sum_{k=0}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$$

Trig. Identity:

$$a\cos(2\pi f t) - b\sin(2\pi f t) = \sqrt{a^2+b^2}\cos\left(2\pi f t + \arctan\left(\frac{b}{a}\right) + \pi \mathbb{1}_{\{a<0\}}\right)$$

This implies:

1.) $\sum_{k=0}^{\infty} a_k \cos(2\pi k f t) - b_k \sin(2\pi k f t)$ can represent any periodic function

2.) Encode into complex numbers:

Let c_k be complex with $c_k = a_k + i b_k$

$$(1) = \sum_{k=0}^{\infty} \mathrm{Re}\{c_k\}\cos(2\pi k f t) - \mathrm{Im}\{c_k\}\sin(2\pi k f t) = \sum_{k=0}^{\infty} |c_k|\cos(2\pi k f t + \measuredangle c_k)$$

where $\measuredangle c_k$ is the phase of c_k (i.e. $c_k = |c_k| e^{i\measuredangle c_k}$)

3.) Euler:

$$\cos(2\pi k f t + \measuredangle c_k) = \frac{1}{2}\left(e^{i2\pi k f t} e^{i\measuredangle c_k} + e^{-i2\pi k f t} e^{-i\measuredangle c_k}\right)$$

$$\Rightarrow (1) = \underbrace{\sum_{k=0}^{\infty} c_k e^{i2\pi k f t}}_{\text{positive frequencies}} + \underbrace{\sum_{k=0}^{\infty} c_k^* e^{-i2\pi k f t}}_{\text{negative frequencies}}$$

relax

Why use complex function?

- Trig. identities are trivial

$$\sin(x)\cos(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\cdot\frac{1}{2}\cdot\left(e^{ix} + e^{-ix}\right)$$

$$= \frac{1}{4i}\left(e^{i2x} - 1 + 1 - e^{-i2x}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{2i}\left(e^{i2x} - e^{-i2x}\right) = \frac{1}{2}\sin(2x)$$

- Multiplication is easy
- Derivatives are easy
- Fourier transform properties are simpler.

Fourier Series: Let $x(t)$ be periodic with period T

Forward: (Analysis) $$c_k = \frac{1}{T}\int_0^T x(\tau)e^{-i2\pi\frac{k}{T}\tau}\,d\tau$$

Backward: (Synthesis) $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi\frac{k}{T}t}$$