Lecture 5

Friday, February 13, 2015
11:10 AM

Transforms:

(CT only) Taylor Series — local behavior at one point.

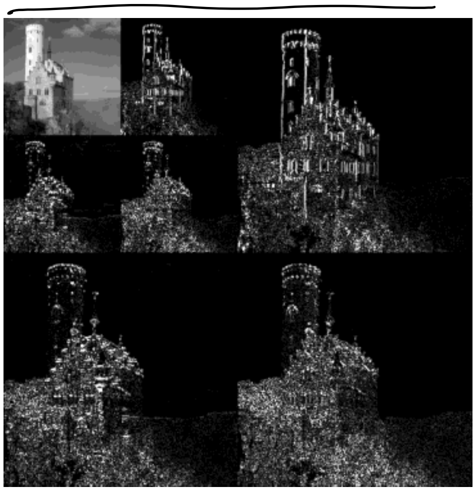
(CT and DT) Wavelet — Separates into pieces of varying location and width

Similar

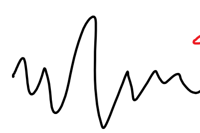
(CT and DT) Fourier — Frequency.

- Short time Fourier tranform — time and frequency

Wavelet Transform:



Start with a mother wavelet:



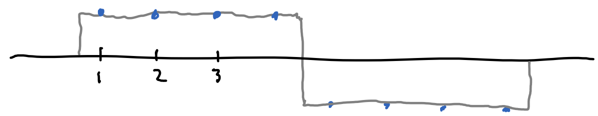
Different transform for diff. mother wavelet

Separate signal into sum of wavelets, each with a different time shift and scale

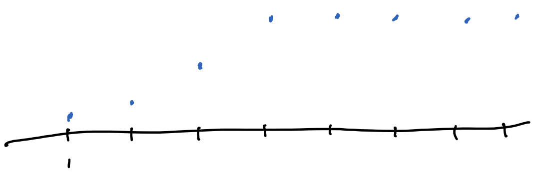
Example: Discrete-time Haar wavelet:

Haar mother wavelet: $(1 \;\; 1 \;\; 1 \;\; 1 \;\; -1 \;\; -1 \;\; -1 \;\; -1)$

Haar mother wavelet: $(1\ 1\ 1\ 1\ -1\ -1\ -1\ -1)$



Signal:
$(1\ 2\ 5\ 8\ 8\ 8\ 8\ 8)$



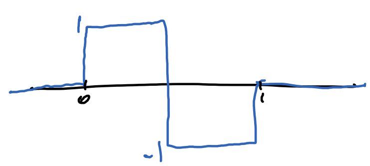
Only need 8 wavelets:

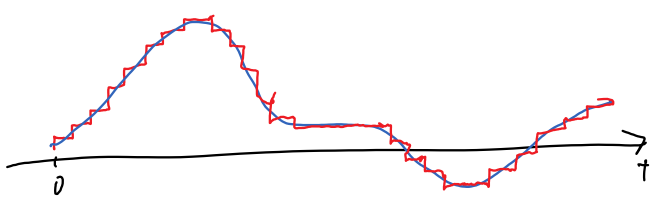
Wavelets	Coefficients:
$(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$	6
$(1\ 1\ 1\ 1\ -1\ -1\ -1\ -1)$	-2
$(1\ 1\ -1\ -1\ 0\ 0\ 0\ 0)$	-2.5
$(0\ 0\ 0\ 0\ 1\ 1\ -1\ -1)$	0
$(1\ -1\ 0\ 0\ 0\ 0\ 0\ 0)$	-0.5
$(0\ 0\ 1\ -1\ 0\ 0\ 0\ 0)$	-1.5
$(0\ 0\ 0\ 0\ 1\ -1\ 0\ 0)$	0
$(0\ 0\ 0\ 0\ 0\ 0\ 1\ -1)$	0

Zeros

Check: 1 2 5 8 8 8 8 8

CT Haar Wavelet: Mother





Uses of transforms:

- Compression
  https://unix4lyfe.org/haar/
- Pattern Recognition / Machine Learning
- Denoising

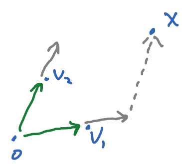
$\Big\}$ Sparsity

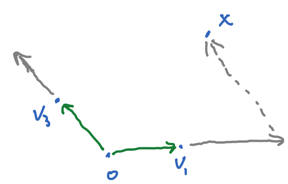
We will see other reasons why Fourier is special.

In general, splitting into a linear combination of basis vectors involves interactions between basis vectors

(Mathematically, matrix inverse)

Consider 2-D:





Orthonormal basis:

Decomposition is very easy (avoids the above)

$$x[n] = a_1 v_1[n] + a_2 v_2[n] + \dots = \sum_k a_k v_k[n]$$

If $\{v_k[n]\}$ are orthonormal

$$\langle x[n], v_k[n] \rangle = \left\langle \sum_j a_j v_j[n], v_k[n] \right\rangle = \sum_j a_j \langle v_j, v_k \rangle$$

$$= \sum_j a_j \mathbb{1}_{j=k}$$

$$= a_k$$

Formula: $a_k = \langle x[n], v_k[n] \rangle$

Formula:

$$a_K = \langle x[n], v_K[n] \rangle$$

$$x[n] = \sum_K a_K \, v_K[n]$$

Fourier Series:

Continuous-time: Let $v_K(t) = e^{i 2\pi \frac{K}{T} t}$

The $\{v_K(t)\}$ are periodic with period T.

Use the periodic inner product

Claim: $\{v_K(t)\}$ are orthonormal

$$\langle v_K(t), v_j(t) \rangle_T = \frac{1}{T} \int_0^T e^{i 2\pi \frac{K}{T} t} \left(e^{i 2\pi \frac{j}{T} t} \right)^* dt$$

$$= \frac{1}{T} \int_0^T e^{i 2\pi \frac{K-j}{T} t} \, dt$$

$$= \begin{cases} \frac{1}{T} \int_0^T dt = 1 & K = j \\ \frac{1}{T} \left[\frac{1}{i 2\pi \frac{K-j}{T}} e^{i 2\pi \frac{K-j}{T} t} \right]_0^T = 0, & K \neq j \end{cases}$$

Discrete-time: $v_K[n] = e^{i 2\pi \frac{K}{N} n}$ for some integer N.

- period N

Claim: $\{v_K[n]\}_{K=0}^{N-1}$ are orthonormal:

Proof:

$$\langle v_K[n], v_j[n] \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} e^{i 2\pi \frac{K-j}{N} n}$$

$$\langle v_k[n], v_i[n] \rangle_N = \frac{1}{N} \sum_{n=0} e$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(e^{i2\pi \frac{k-i}{N}}\right)^n \longleftarrow \text{Geometric sum}$$

$$\boxed{\text{Geometric series: If } a \neq 1, \ \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \Rightarrow \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{ if } |a| < 1}$$

Therefore:

Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, v_k(t)$$

Analysis

$$a_k = \langle x(t), v_k(t) \rangle_T = \frac{1}{T} \int_0^T x(t) \, e^{-i2\pi \frac{k}{T} t} \, dt$$

The Fourier series emerges by plugging in the complex exponential basis to the general expression.