Lecture 6

Wednesday, February 18, 2015   8:17 AM

What is fft?   Fast Fourier Transform

You've used in Matlab

Finite Duration Signals:

- Equivalent to periodic signal.

1.) Convert $x(t)$ with period T to finite duration.

- Keep only one period

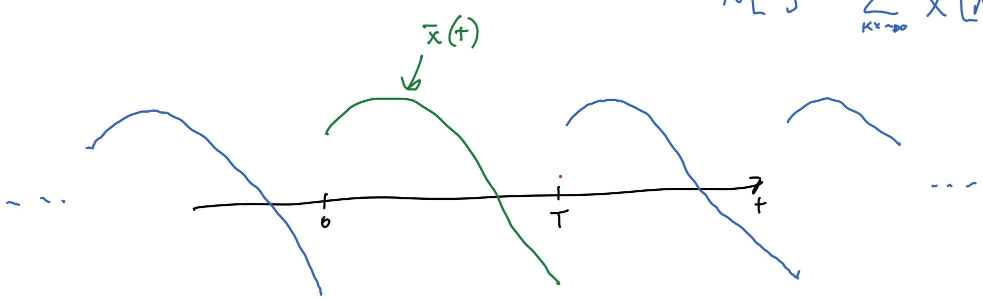
$$\bar{x}(t) = \begin{cases} x(t), & t \in [0,T) \\ 0, & \text{else} \end{cases}$$

$$\bar{x}[n] = \begin{cases} x[n], & n \in \{0, 1, \dots, N-1\} \\ 0, & \text{else} \end{cases}$$

2.) Convert finite duration of length T to periodic:

- Periodic extension: $x(t) = \sum_{K=-\infty}^{\infty} \bar{x}(t - KT)$

$$x[n] = \sum_{K=-\infty}^{\infty} \bar{x}[n - KN]$$



Fourier Series:

CTFS: $x(t) \longrightarrow \{a_K\}_{K=-\infty}^{\infty}, \; T$

$x[n] \longrightarrow \{a_K\}_{K=-\infty}^{\infty}, \; N$

Must Know the period in order to recover x.

In fact, only $K = 0, \dots, N-1$ needed. (DT only)

"Discrete Fourier Transform"   DFT: $\vec{x} \longrightarrow \vec{a}$

Vectors of same length

Length encodes the period.

$$\vec{x} = (x[0], x[1], \ldots, x[N-1])$$

DFT is essentially DTFS aside from scaling

Consider DTFS for $N=4$:

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-i2\pi \frac{k}{4} n}$$

$$a_0 = \frac{1}{4}\left(x[0] + x[1] + x[2] + x[3]\right)$$
$$a_1 = \frac{1}{4}\left(x[0] + -i\,x[1] - x[2] + i\,x[3]\right)$$
$$a_2 = \frac{1}{4}\left(x[0] - x[1] + x[2] - x[3]\right)$$
$$a_3 = \frac{1}{4}\left(x[0] + i\,x[1] - x[2] - i\,x[3]\right)$$

Example   Period 4

$$x[n] = \begin{cases} 1 & n=0 \\ 1 & n=1 \\ 1 & n=2 \\ 0 & n=3 \end{cases}$$

$a_0 = 3/4$

$a_1 = -i/4$

$a_2 = 1/4$

$a_3 = i/4$

Matrix form:

$$a = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} x$$

Call this $[DTFS_4]$

DTFS is a matrix multiply

In general, define $W_N = e^{i2\pi/N}$

$$[C\text{-}Exp_N] = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & & \\ \vdots & & & & \end{bmatrix}$$

An $N \times N$ matrix

Notice $[C\text{-}Exp_N]^T = [C\text{-}Exp_N]$

Fourier Series:

$$[DTFS_N] = \frac{1}{N}[C\text{-}Exp_N]^* . \qquad [DTFS_N]^{-1} = [C\text{-}Exp_N]$$

difference

Only difference

Discrete Fourier Transform:

Definition of DFT:

$$[DFT_N] = [C\text{-}Exp_N]^*, \qquad [DFT_N]^{-1} = \frac{1}{N}[C\text{-}Exp_N]$$

> Linear Algebra Note:
>
> $\frac{1}{\sqrt{N}}[C\text{-}Exp_N]$ is unitary (inverse is conjugate transpose)
>
> - This change of basis diagonalizes toeplitz matrices

Fast Fourier Transform:

DFT is $N \times N$ matrix multiplication.

N^2 complexity

FFT is an algorithm to do this in $N \log N$ complexity

- Cooley-Tukey (1965)
- Huge impact on the use in digital signal processing (DSP)

$[DFT_N]$ factorizes:

$$[DFT_8] = \left[\begin{array}{cccc|cccc} 1 & & & 0 & 1 & & & 0 \\ & 1 & & & & W_8^1 & & \\ & & 1 & & & & W_8^2 & \\ 0 & & & 1 & 0 & & & W_8^3 \\ \hline 1 & & & 0 & -1 & & & 0 \\ & 1 & & & & -W_8^1 & & \\ & & 1 & & & & -W_8^2 & \\ 0 & & & 1 & 0 & & & -W_8^3 \end{array}\right] \left[\begin{array}{c|c} DFT_4 & 0 \\ \hline 0 & DFT_4 \end{array}\right] \left[\begin{array}{c|c} \begin{matrix} 1000 \\ 0010 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 1000 \\ 0010 \end{matrix} \\ \hline \begin{matrix} 0100 \\ 0001 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 0100 \\ 0001 \end{matrix} \end{array}\right]$$

Aliasing: Discrete-time only.

Two sinusoids can be the same in discrete-time.

$x[n] = \cos(2\pi n)$, $\quad n \in \mathbb{Z}$

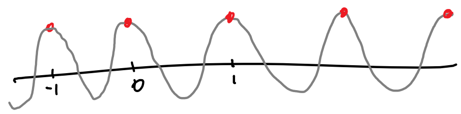
$x[0] = 1$

$x[1] = 1$

$x[2] = 1$

$\vdots$

$\cos(2\pi n) = 1$



http://demonstrations.wolfram.com/AliasingInTimeSeriesAnalysis/

$$\boxed{e^{i2\pi f_1 n} = e^{i2\pi f_2 n} \quad \text{if } f_1 - f_2 \text{ is an integer}}$$

signals

$\forall$ integer n

Verify: $\dfrac{e^{i2\pi f_1 n}}{e^{i2\pi f_2 n}} = e^{i2\pi (f_1 - f_2) n} = 1$

fftshift:

Revisit example of DTFS:

$$x[n] = \begin{cases} 1 & n=0 \\ 1 & n=1 \\ 1 & n=2 \\ 0 & n=3 \end{cases}$$

frequency $f = \frac{k}{N}$

$a_{-2} = a_2 \longleftarrow f = \frac{-2}{4}$ (aliasing)

$a_{-1} = a_3 \longleftarrow f = \frac{-1}{4}$ (aliasing)

$a_0 = \frac{3}{4} \longleftarrow f = 0$

$a_1 = -i/4 \longleftarrow f = \frac{1}{4}$

$a_2 = 1/4 \longleftarrow f = \frac{2}{4}$

$a_3 = i/4 \longleftarrow f = \frac{3}{4}$

$$a_4 = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-i2\pi \frac{4}{4} n} = \frac{1}{4}\sum_{n=0}^{3} x[n] e^{-i2\pi 0 n} = a_0$$

$N = 4$: $\quad a_k = \cdots = a_{k-N}$ due to aliasing

a_k is periodic with period N