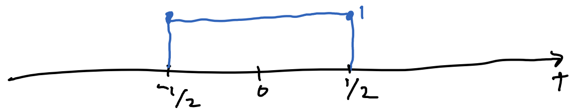
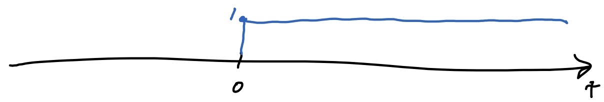
Fourier Transform Example:

Useful functions:

Rect function: $\mathrm{rect}(t) = \begin{cases} 1, & |t| \le \frac{1}{2} \\ 0, & \text{else} \end{cases}$

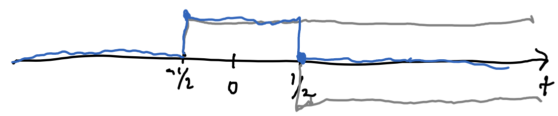


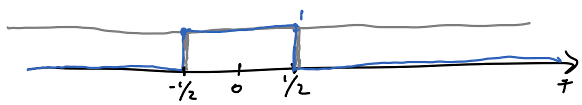
Unit step functions: $u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & \text{else} \end{cases}$



Notice: $\mathrm{rect}(t) = u(t + \frac{1}{2}) - u(t - \frac{1}{2})$

$= u(t + \frac{1}{2})\, u(-t + \frac{1}{2})$





Fourier Transform of rect(t):

$$X(f) = \int_{-\infty}^{\infty} \mathrm{rect}(t)\, e^{-i2\pi f t}\, dt$$

$$= \int_{-1/2}^{1/2} e^{-i2\pi f t}\, dt$$

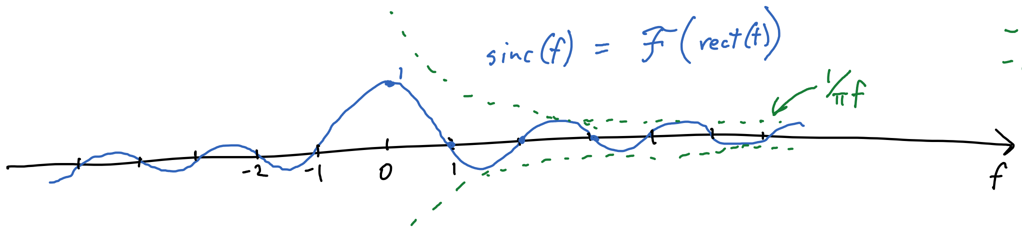
$$= \begin{cases} \frac{-1}{i2\pi f} \left[e^{-i2\pi f t} \right]_{t=-1/2}^{1/2}, & f \ne 0 \\ 1, & f = 0 \end{cases}$$

$$= \begin{cases} \frac{-1}{i2\pi f} \left(e^{-i\pi f} - e^{i\pi f} \right), & f \ne 0 \\ 1, & f = 0 \end{cases}$$

$$= \begin{cases} \dfrac{\sin(\pi f)}{\pi f}, & f \neq 0 \\ 1, & f = 0 \end{cases} = \frac{\sin(\pi f)}{\pi f} \triangleq \operatorname{sinc}(f)$$

sinc:
- even
- value one at center
- zero at other integers
- decays as $1/\pi f$

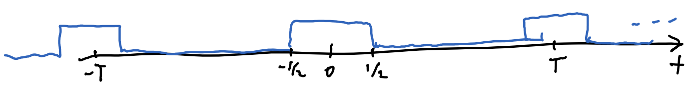
$$\operatorname{sinc}(f) = \mathcal{F}(\operatorname{rect}(t))$$



Fourier Series Calculation:

Let $T > 1$

Let $x(t)$ be the periodic extension of $\operatorname{rect}(t)$



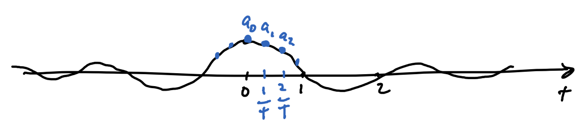
CTFS: $$a_K = \frac{1}{T}\int_0^T x(t)\, e^{-i2\pi \frac{K}{T} t}\, dt$$

any one period

$$= \frac{1}{T}\int_{-T/2}^{T/2} \cancel{x(t)}\ \operatorname{rect}(t)\, e^{-i2\pi \frac{K}{T} t}\, dt$$

$$= \frac{1}{T} X\left(\frac{K}{T}\right)$$

$$= \frac{1}{T} \operatorname{sinc}\left(\frac{K}{T}\right)$$



Recover $$x(t) = \sum_K a_K e^{i2\pi \frac{K}{T} t}$$

$$= \frac{1}{T}\sum_K \operatorname{sinc}\left(\frac{K}{T}\right) e^{i2\pi \left(\frac{K}{T}\right) t}$$

$$\rightarrow \boxed{\int_{-\infty}^{\infty} \operatorname{sinc}(f)\, e^{i2\pi f t}\, df}$$

Inverse Fourier Transform: $$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f t}\, df.$$

Formulas: CTFT:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{i2\pi f t}\, df \quad \leftarrow \text{inverse}$$

Formulas: CTFT:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df$$

inverse (synthesis)

DTFT:

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-i2\pi fn}$$

$$x[n] = \int_{-1/2}^{1/2} X(f) e^{i2\pi fn}$$

(one period)

f is not discrete

In DT: X(f) has period one due to aliasing

Properties: **Duality:**

Forward and Inverse CTFT are (almost) the same.

Let $y(f) = \mathcal{F}(x(t))$

then $y(-t) = \mathcal{F}^{-1}(x(f))$

$$x \xrightarrow{\mathcal{F}} X \Longrightarrow X \xrightarrow{\mathcal{F}} x \text{ flipped horizontally}$$

Check:

$$z(t) = \int_{-\infty}^{\infty} x(f) e^{i2\pi ft} df = \mathcal{F}^{-1}(x(f))$$

$$z(-\tau) = \int_{-\infty}^{\infty} x(f) e^{-i2\pi f\tau} df$$

$$= \int_{-\infty}^{\infty} x(t) e^{-i2\pi t\tau} dt$$

$$= \mathcal{F}(x(t))$$

Application of duality:

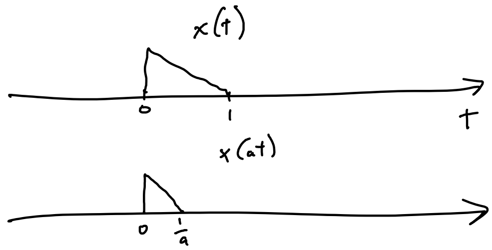
$$\text{rect}(t) \xrightarrow{\mathcal{F}} \text{sinc}(f)$$

$$\text{sinc}(t) \xrightarrow{\mathcal{F}} \text{rect}(-t) = \text{rect}(t)$$

Time scaling property:

$x(at)$

$\uparrow$ real number



Assume $X(f)$ is CTFT of $x(t)$

$$\mathcal{F}\left(x(at)\right) = \int_{-\infty}^{\infty} x(at)\, e^{-i2\pi f t}\, dt$$

let $\tau = at$

$$= \frac{1}{|a|}\int_{-\infty}^{\infty} x(\tau)\, e^{-i2\pi \left(\frac{f}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Linearity:

Two requirements:

- Superposition Property $\quad G(x+y) = G(x) + G(y)$
- Scaling Property $\quad G(ax) = a\,G(x)$

Check FT:

$$\mathcal{F}\left(a\,x(t) + y(t)\right) = \int_{-\infty}^{\infty}\left(a x(t) + y(t)\right) e^{-i2\pi f t}\, dt$$

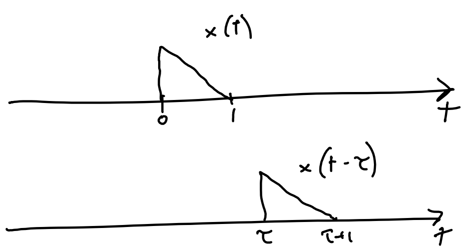
$$= a\int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt + \int_{-\infty}^{\infty} y(t)\, e^{-i2\pi f t}\, dt$$

$$= a\,X(f) + Y(f)$$

Holds for DT as well.

Time-shift Property:

$x(t-\tau)$



$$\mathcal{F}(x(t-\tau)) = \int_{-\infty}^{\infty} x(t-\tau)\, e^{-i2\pi f t}\, dt$$

$u = t - \tau$

$$= \int_{-\infty}^{\infty} x(u)\, e^{-i2\pi f(u+\tau)}\, du$$

$$= e^{-i2\pi f\tau} \int_{-\infty}^{\infty} x(u)\, e^{-i2\pi f u}\, du$$

$$= \boxed{e^{-i2\pi f\tau}}\, X(f)$$

Magnitude 1
Phase $-2\pi f\tau$

$\Rightarrow$ Magnitude doesn't change

Phase becomes $\angle X(f) - 2\pi\tau f$

Holds for DT with integer shifts.