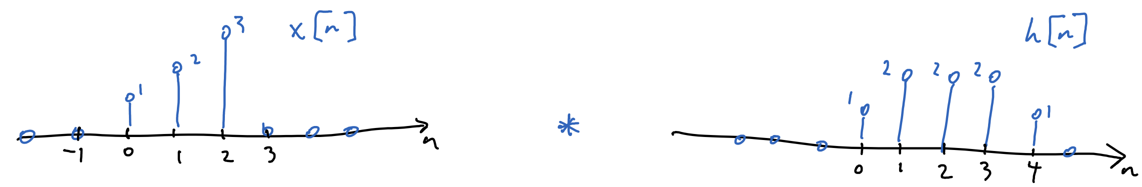
Convolution

Discrete-time: $\underbrace{x[n]}\ast\underbrace{h[n]} = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

Two signals

One signal (function of n)

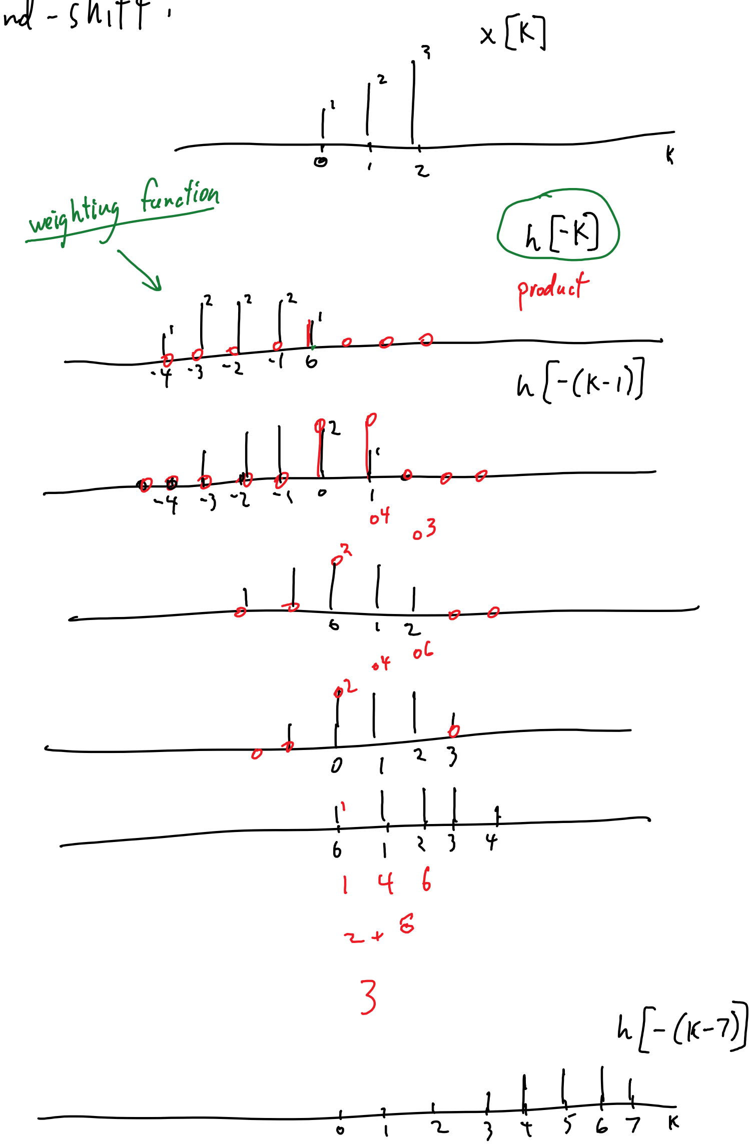


Calculate for each n seperately:

Flip-and-shift:

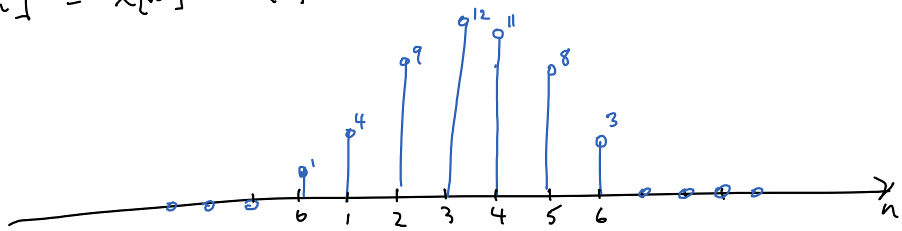
$h[n-k] = h[-(k-n)]$

Must do negative values of n as well. Zero in this case



	Output:
$n=0$:	1
$n=1$:	4
$n=2$:	9
$n=3$:	12
$n=4$:	11
$n=5$:	8
$n=6$:	3
$n=7$:	0
	⋮

$y[n] = x[n] * h[n]$



Convolution is linear (for fixed h)

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{Convolution}\\ *h[n]\end{array}} \xrightarrow{y[n]}$$

$$(a\, x_1[n] + x_2[n]) * h[n] = a\,(x_1[n] * h[n]) + (x_2[n] * h[n])$$

$$\parallel \quad \sum_{k=-\infty}^{\infty} (a\, x_1[k] + x_2[k])\, h[n-k]$$

$$= a\left(\sum_k a\, x_1[k]\, h[n-k]\right) + \left(\sum_k x_2[k]\, h[n-k]\right)$$

Polynomial Multiplication is convolution:

$0z^{-1}$ $\quad (1 - 2z + z^2)(5 + 4z)$

z^0 term $\longrightarrow$ $4z + 5$

z^1 term $\longrightarrow$ $4z + 5$

z^2 term $\longrightarrow$ $4z + 5$

Vector

$(1, -2, 1)$

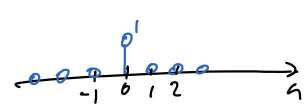
$(5, 4)$

$(1, -2, 1) * (5, 4) =$ resulting polynomial.

Useful discrete-time functions:

Kronecker delta function:

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{else} \end{cases}$$



Unit step function: $u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{else} \end{cases}$

We care about each point : $u[0] = 1$

Delta function property:

<u>Sifting</u>

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n] = x[0]$$

$$\sum_{n=-\infty}^{\infty} x[n]\,\delta[n-N] = x[N]$$

$$x[n]\,\delta[n-N] = x[N]\,\delta[n-N]$$

Interpret a signal as composed of delta functions.

$$x[n] = \sum_{K=\infty}^{\infty} x[K]\,\delta[n-K]$$

<u>Convolution with $\delta[n]$:</u>

$$h[n] * \delta[n] = \sum_K h[K]\,\delta[n-K] = h[n]$$

$$\delta[n] * h[n] = h[n]$$

Convolution commutes:

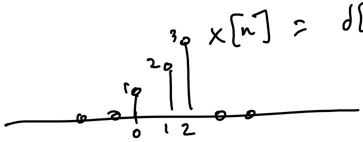
$$x[n] * h[n] = h[n] * x[n]$$

$$= \sum_K x[K]\,h[n-K] = \sum_m x[n-m]\,h[m]$$

$m = n-K$

<u>Alternate approach to convolution:</u>

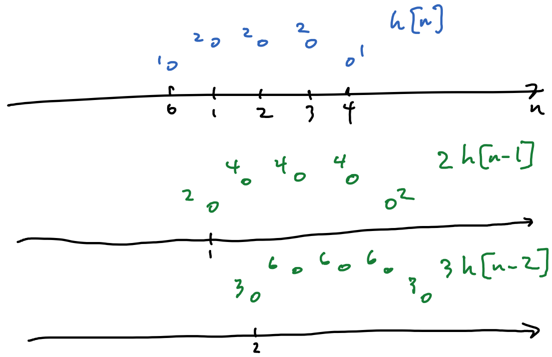
$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] \qquad h[n]$$



$$x[n] * h[n] = \delta[n] * h[n] + 2\,\delta[n-1] * h[n] + 3\,\delta[n-2] * h[n]$$

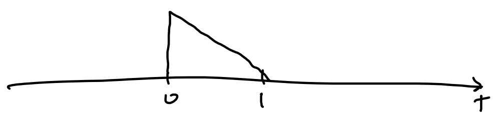
$$= h[n] + 2\,h[n-1] + 3\,h[n-2]$$

Convolution is time-invariant:



Time shift and Time scale:

$x(t)$

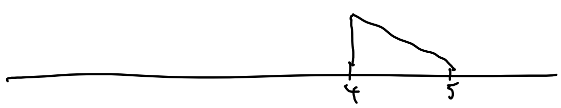


$x(-2t-4)$

1.) Shift right by 4 ← outside first

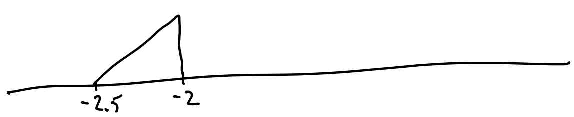
2.) Scale by -2

1.) $z(t) = x(t-4)$



2.) $x(-2t-4)$

$z(-2t)$



Alternative: Scale first then shift

$x(-2t-4) = x(-2(t+2))$

1.) Scale by -2
2.) Shift left by 2

$\mathcal{F}(x(-2t-4))$

Start with $\mathcal{F}(x(t)) = X(f)$

$$\mathcal{F}(x(t-4)) = e^{-i2\pi 4f} X(f)$$

$$\mathcal{F}(x(-2t-4)) = \frac{1}{2} e^{-i2\pi 4\left(\frac{f}{-2}\right)} X\left(\frac{f}{-2}\right)$$

$$= \frac{1}{2} e^{i4\pi f} X\left(\frac{f}{-2}\right)$$