

ELE 201 – Problem Set 4

Due March 27

1. Prove Parseval's Theorem for the continuous-time and discrete-time Fourier transforms. That is, show that the following two identities are valid:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$
$$\sum_{k=-\infty}^{\infty} |x[k]|^2 = \int_{-1/2}^{1/2} |X(f)|^2 df$$

Use the following steps. First, define $y(t) = x(t) * x^*(-t)$ (this is the autocorrelation function). Show that $y(0)$ is the energy of $x(t)$.

Next, derive $Y(f)$ in terms of $X(f)$.

Finally, use the synthesis equation (inverse Fourier transform) to express $y(0)$ in terms of $Y(f)$.

2. Calculate the DTFT of $\text{rect}\left[\frac{n}{N}\right] = u\left[n + \frac{N}{2}\right]u\left[-n + \frac{N}{2}\right]$, where N is even, using the geometric series. A change of variables ($m = k + N/2$) will help. Simplify the answer to be expressed only with trigonometric functions (no complex exponentials) using Euler's identity both in the numerator and denominator. Compare this to the CTFT of the continuous-time rect.
3. What is the energy of $\text{sinc}^2(t)$?
4. Suppose an LTI system has impulse response $h(t) = \frac{1}{3}\text{rect}(t/4)$. Can the output have more energy than the input? Can the output have less energy than the input?
5. Define the following signal:

$$x(t) = \begin{cases} t+1, & -1 \leq t \leq 1 \\ 2, & 1 \leq t \leq 3 \\ 5-t, & 3 \leq t \leq 5 \\ 0, & \text{else} \end{cases}$$

Calculate the following, where $X(f)$ is the Fourier transform of $x(t)$. (Hint: You don't need to calculate $X(f)$ for any of these calculations. Start by plotting $x(t)$.)

- a. Find the phase of $X(f)$ (as a function of f).
 - b. Find $X(0)$.
 - c. Find $\int_{-\infty}^{\infty} X(f) df$.
 - d. Find $\int_{-\infty}^{\infty} X(f) \text{sinc } f e^{i4\pi f} df$.
 - e. Find $\int_{-\infty}^{\infty} |X(f)|^2 df$.
 - f. Sketch the inverse Fourier transform of $\text{Re}\{X(f)\}$.
6. **Sampling:** Suppose $x(t)$ is real and has a non-zero Fourier transform only on the set of frequencies 5-8Hz. Draw an illustration of the spectrum of $x(t)$. Also draw an illustration of the spectrum when $x(t)$ is sampled at 16Hz. What is the lowest sampling rate from which the signal can still be recovered?