ELE 301, Fall 2010
Laboratory No. 7
Stability and Root Locus Plots

1 Background

1.1 Transfer Function of a Circuit

Model resistors, capacitors and inductors as linear time-invariant components. To obtain the transfer function of a circuit made up of these components, we determine each individual component's current to voltage transfer function (impedance).

The current-voltage relationship for a capacitance C is $I(t) = C(dV(t)/dt)$. So when $V(t) = Ve^{st}$, $I(t) = CsVe^{st} = Ie^{st}$. Hence $V = Z_C I$, where $Z_C = 1/(sC)$.

The current-voltage relationship for inductance is $V(t) = L(dI(t)/dt)$. As above consider $V(t) = Ve^{st}$. Then $I(t) = LsVe^{st} = Ie^{st}$. So $V = Z_L I$, where $Z_L = sL$. You can apply the same analysis to a resistance R to obtain $V = IR$.

So for inputs of the form e^{st}, circuit components have voltage-current relationships that look like Ohm's law, except that the impedances are now complex:

$$Z_R = R \qquad Z_C = 1/(sC) \qquad Z_L = sL$$

All the rules of resistor combination in series and in parallel apply to impedances, so sinusoidal circuit analysis becomes very easy.

As an example, we analyze the RC circuits in Figure 1. We use basic circuit laws to calculate

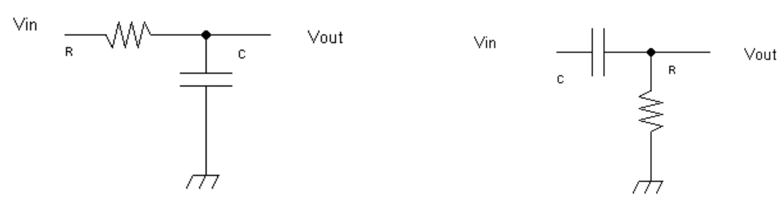

Figure 1: Two RC circuits.

$v_{out} = H(s)v_{in}$. Each circuit is a voltage divider. For the circuit on the left:

$$v_{out} = v_{in}\frac{1/sC}{R + 1/sC} = v_{in}\frac{1}{1 + sRC}$$

Hence

$$H_1(s) = \frac{1}{1 + sCR}$$

For the circuit on the right:

$$v_{out} = v_{in}\frac{R}{R + 1/sC} = v_{in}\frac{sCR}{1 + sRC}$$

Hence

$$H_2(s) = \frac{sCR}{1 + sCR}$$

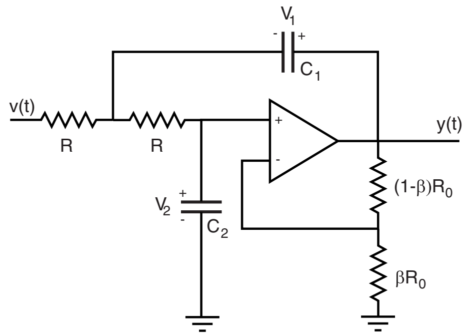

Figure 2: Active low pass filter: Sallen-Key circuit.

1.2 A Second Order System

The circuit shown in Figure 2 is a variation on the Sallen-Key circuit. The circuit contains two capacitors and gives rise to a second order differential equation. For this reason, it is called a second order system. Assuming that R_0 is very large and that $C_1 = C_2$, the circuit transfer function from x to y is:

$$H(s) = \frac{\tau^2/\beta}{s^2 + (3 - 1/\beta)\tau s + \tau^2} \tag{1}$$

where $\tau = 1/(RC)$ and $0 < \beta \leq 1$.

We can parameterize the transfer function of a second order system as follows:

$$H(s) = G\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{2}$$

The two parameters ζ (zeta) and ω_n are called the *damping factor* and the *natural frequency* of the system, respectively, and G is a gain factor.

1.3 Questions

1. Find the gain G, natural frequency ω_n, and damping factor ζ of the transfer function $H(s)$ in (2) as functions of β and τ.

2. For what value, if any, of β is the circuit critically damped?

3. Is the circuit underdamped for any the realizable values of *beta*? If so for what values?

4. For what range of $\beta \in (0, 1]$ is this circuit stable and why?

5. Derive the transfer function (2) of the Sallen-Key circuit in Figure 2. You can attach this.

2 Lab Procedure

2.1 Plotting Root Locus

A root locus of a transfer function is a plot of the poles of the transfer function (roots of the denominator polynomial) as a scalar parameter is varied. The roots of a polynomial are continuous functions of its coefficients and the coefficients of the denominator polynomial of $H(s)$ are continuous functions of β. Hence as β is varied the roots trace out continuous curves in the complex plane.

Create a root locus plot of the poles of the transfer function $H(s)$ of the Sallen-Key circuit as β increases from 0.25 to 1. Do this by creating a loop that cycles through the desired values of β in the range $[0.25, 1]$ and for each value:

1. creates a vector of the coefficients of the denominator polynomial `den` of $H(s)$ (highest to lowest powers of s).
2. calls the built-in MATLAB m-file `roots` to compute the roots of the polynomial.
3. plots the position of the roots on the complex plane (use `hold on` and `hold off`).

To plot a 2-D points with (vectors of) coordinates (x, y) you can use:

```
plot(x,y,'r.','MarkerSize',14);
```

Use the grid command:

```
sgrid
```

after the end of the loop to add a grid that indicates circles of constant ω_n and radial lines of constant ζ . You can add text to the plot with the command:

```
text(xp,yp,'Root locus for β=1,...,0.25');
```

Where (xp, yp) is the desired start position for the text. Include such annotation on your plot. Make sure you also label all axis.

What does the root locus plot confirm about the stability of this circuit?

2.2 A Simple Feedback Loop

Now that we have a program to plot a root locus we will use it to explore the stability of the simple negative feedback system shown in Figure 3.

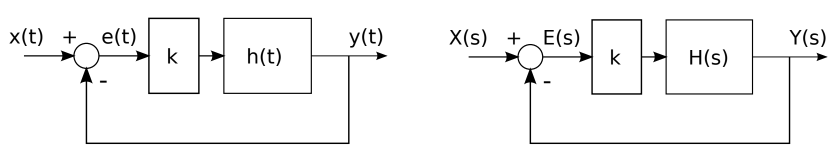

Figure 3: Proportional error feedback regulation. (l) time domain, (r) Laplace domain.

$H(s)$ is the transfer function of the open loop system. It is assumed to be a rational function. The closed loop transfer function from x to y is:

$$F(s) = \frac{kH(s)}{1 + kH(s)} = \frac{p(s)}{q(s)}$$

where $p(s)$ and $q(s)$ are polynomials in s. The stability of the system requires that the roots of $q(s)$ be in the open left half of the complex plane (see Laplace transform notes).

For each of the transfer functions $H(s)$ in the list below:

- Compute the closed loop transfer function $F(s)$.

- Plot a root locus of the poles of the closed loop system as k varies.
- Interpret the root locus (explain what is happening).
- Use the root locus plot to find the largest k (if it exists) such that the poles are stable and either real of have a damping factor of approximately 0.75.

1. $H(s) = \frac{1}{s-1}$ (this system is unstable).
2. $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ with $\zeta = 0.85$ and $\omega_n = 2$.
3. $H(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$ with $\zeta = 0.85$ and $\omega_n = 2$.
4. $H(s)$ as in 3. except with $\zeta = 0.95$ (almost critically damped).
5. $H(s)$ as in 3. except with $\zeta = 0.65$ (lightly damped).

Attach all your plots and code to the lab handout prior to handing it in.