Tuesday, October 18, 2011
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The DFT (half the story)

10/18/11

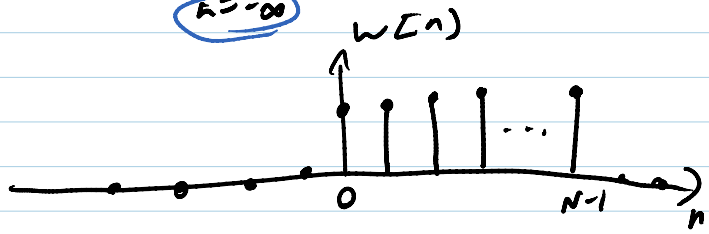
DFT (Discrete Fourier Transform)
of a finite duration signal (discrete time)

$y[0] \ldots y[N-1]$ length N

$$\hat{y}[k] = \sum_{n=0}^{N-1} y[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn}$$

Discrete time signal $x[n]$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$$



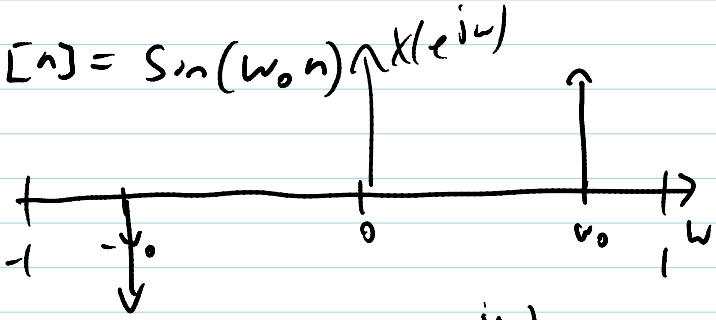
$y[n] = x[n]\, w[n]$

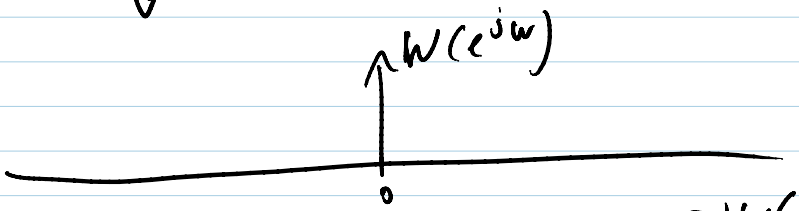
$$Y(e^{jw}) = \sum_{n=0}^{N-1} x[n]\, e^{-jwn} \qquad w = \frac{2\pi k}{N}$$

DFT is sampling $Y(e^{jw})$ at $w = 0, \frac{2\pi}{N}, \frac{2\pi \cdot 2}{N}, \ldots, \frac{2\pi(N-1)}{N}$

Ex

$x[n] = \sin(w_0 n)$

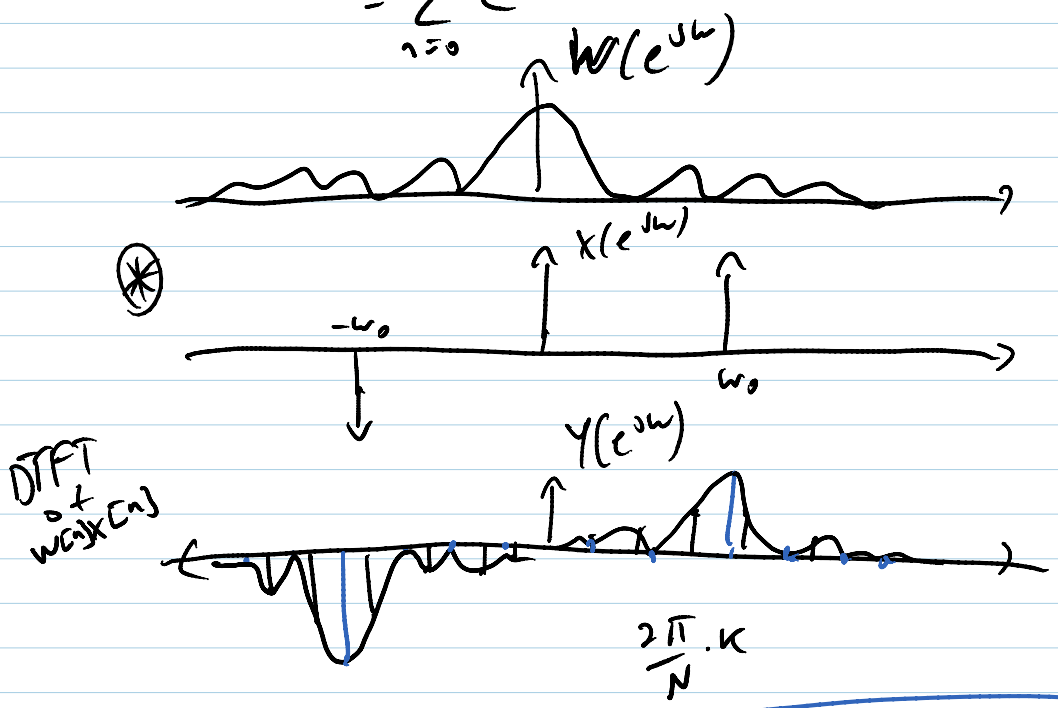




$$W(e^{jw}) = \frac{\sin(w N/2)}{\sin(w/2)}\, e^{-jw(N-1)/2}$$

$N-1$

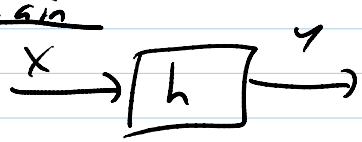
$$= \sum_{n=0}^{N-1} e^{-j\omega n}$$



LTI Systems

- Completely characterized by impulse response

Time Domain



$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Freq. Domain

$$Y(j\omega) = \mathcal{F}\{y(t)\}$$

$$Y(j\omega) = X(j\omega)\, H(j\omega)$$

So H modulates X, the FT of $x(t)$

$H(j\omega)$ is called the <u>Frequency Response</u>

or <u>Transfer Function</u> of the System

- often useful to look at H to see how it affects the different freq. components of the input signal

Let z be a complex number

$$z = a + jb \quad , \quad a, b \in \mathbb{R}$$
$$= |z| e^{j \arg\{z\}} = |z| e^{j \angle z}$$

$$\underline{H(j\omega) = |H(j\omega)| e^{j \angle H(j\omega)}}$$

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

So our eqn from above $Y(j\omega) = H(j\omega) X(j\omega)$

$$|Y(j\omega)| = |H(j\omega)| |X(j\omega)|$$

gain

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

$$\angle (z_1 \cdot z_2) = \angle z_1 + \angle z_2$$

phase

<u>Example</u> $x(t) = e^{j\omega_0 t}$

$$X(j\omega) = 2\pi \delta(\omega - \omega_0)$$

Then
$$Y(j\omega) = H(j\omega) [2\pi \delta(\omega - \omega_0)]$$
$$= H(j\omega_0) [2\pi \delta(\omega - \omega_0)]$$

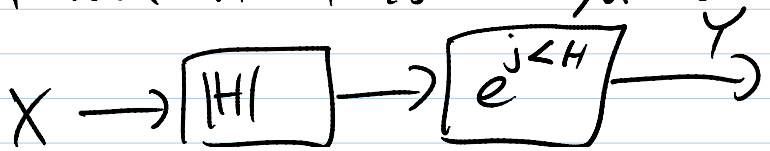
$$y(t) = \mathcal{F}^{-1}\{H(j\omega_0) 2\pi \delta(\omega - \omega_0)\} = H(j\omega_0) e^{j\omega_0 t}$$

$$= |H(j\omega_0)| e^{j(\omega_0 t + \angle H(j\omega_0))}$$

gain phase

<u>end of example</u>

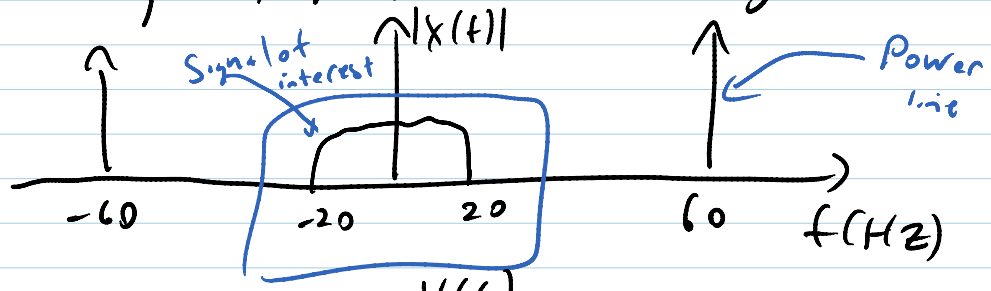
In general can look at H as 2 systems in cascade

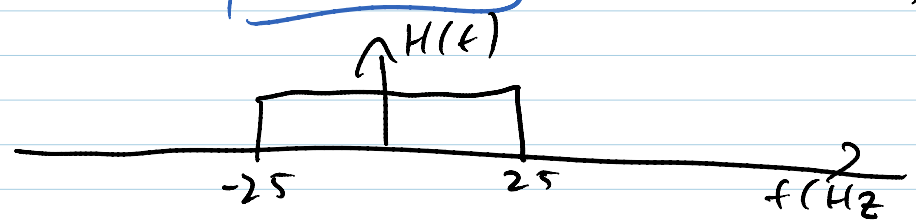


$$X \longrightarrow \boxed{|H|} \longrightarrow \boxed{e^{j\angle H}}$$

Filters

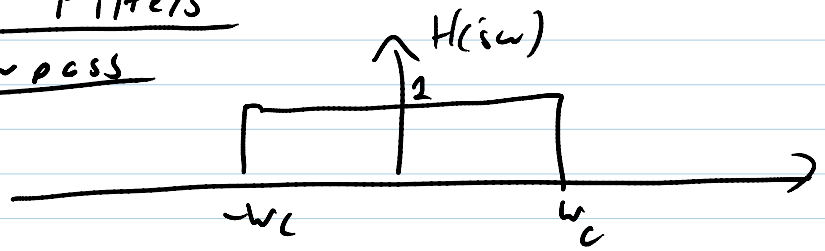
A filter is a system that is used to shape the frequency properties of a signal.



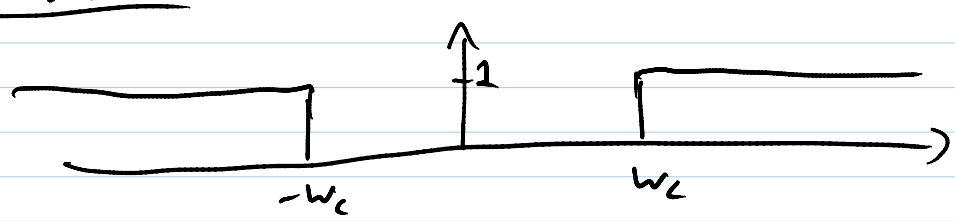


Ideal Filters

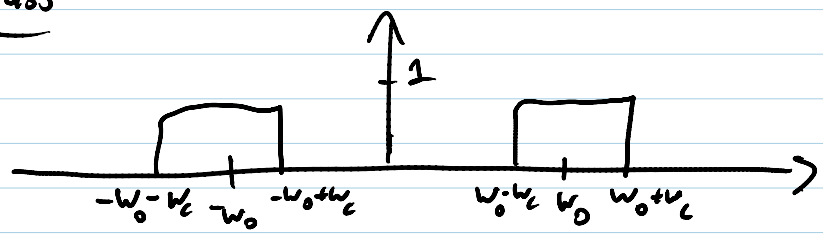
Lowpass



Highpass



Bandpass

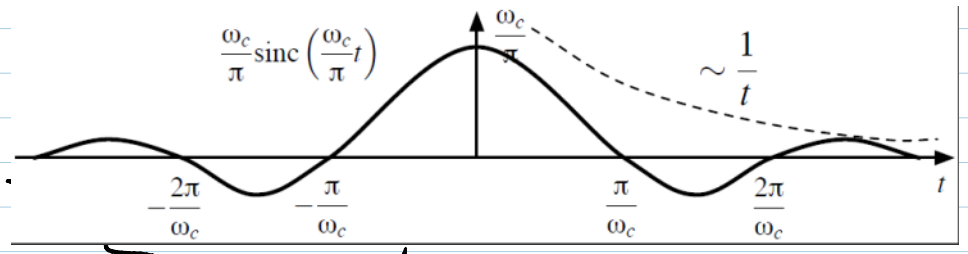


Let's look at the IFT of these filters

Let's look at the IFT of these filters

<u>Lowpass</u> $H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{o.w.} \end{cases}$

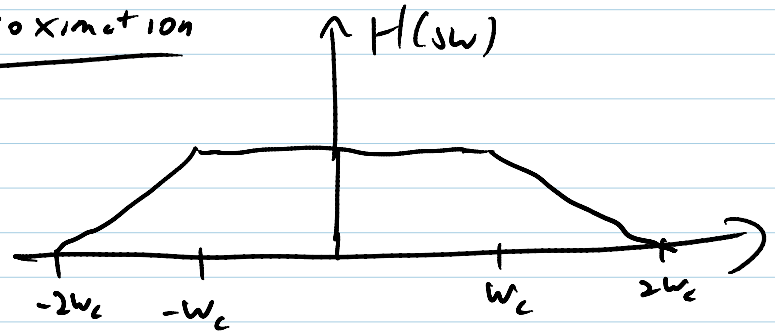
$$h(t) = \frac{\sin \omega_c t}{\pi t} = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

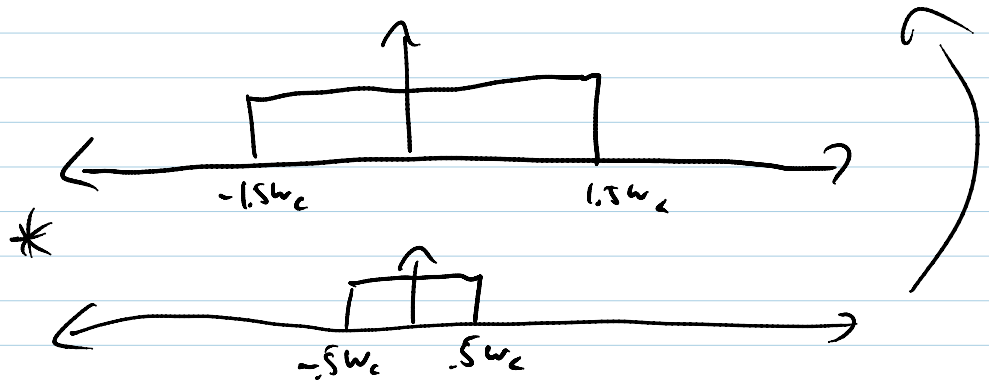


- non causal
- infinite duration
- Decays very slowly $\sim \frac{1}{t}$

$$|h(t)| = \frac{|\sin \omega_c t|}{|\pi t|} \le \frac{1}{\pi |t|} \sim \frac{1}{t}$$

<u>Approximation</u>





$$h(t) = \frac{1.5\omega_c}{\pi} \operatorname{sinc} \frac{1.5\omega_c t}{\pi} \cdot \frac{\omega_c}{2\pi} \operatorname{sinc} \frac{\omega_c t}{2\pi}$$

dies out $\sim \frac{1}{t^2}$ $\Leftarrow$ product of 2 $\frac{1}{t}$'s

$$|\operatorname{sinc} t| = \left|\frac{\sin \pi t}{\pi t}\right| \le \frac{1}{\pi t}$$

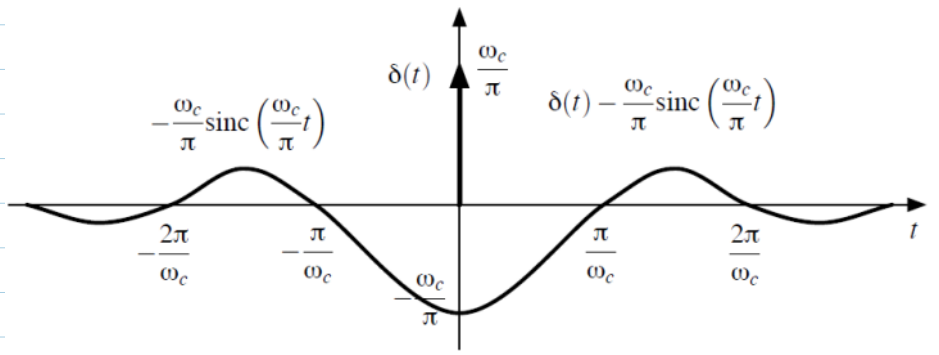
$$\left|\operatorname{Sinc} t\right| = \left|\frac{\sin \pi t}{\pi t}\right| \leq \frac{1}{\pi t}$$

Highpass

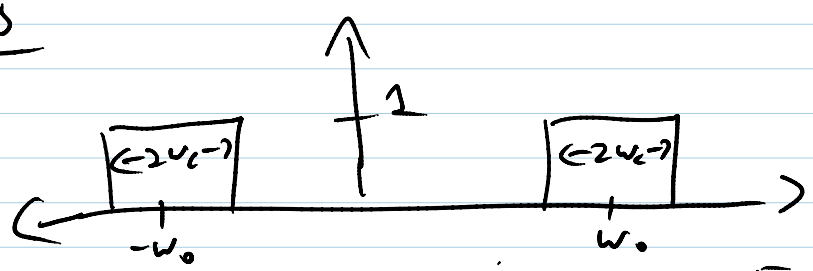
$$H_{hp}(j\omega) = 1 - H_{lp}(j\omega)$$

$$h_{hp}(t) = \delta(t) - \frac{\omega_c}{\pi} \operatorname{Sinc} \frac{\omega_c}{\pi} t$$

not easy to implement in reality



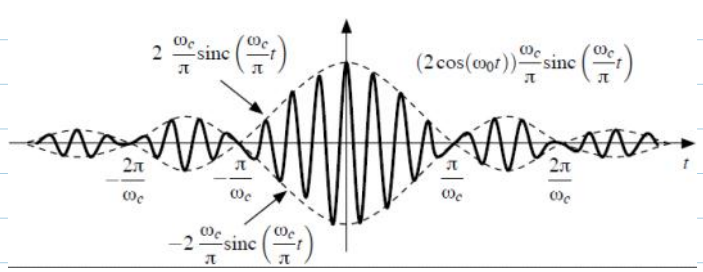
Bandpass



$$h_{BP}(t) = h_{lp}(t) e^{j\omega_0 t} + h_{lp}(t) e^{-j\omega_0 t} = 2h_{lp}(t)\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= 2h_{lp}(t) \cdot \cos(\omega_0 t) = 2\cos(\omega_0 t) \frac{\omega_c}{\pi} \operatorname{Sinc} \frac{\omega_c}{\pi} t$$

Cos in the envelope of a sinc.



Phase

$$h(t-t_0) \longleftrightarrow H(j\omega) e^{-j\omega t_0}$$

$$h_{lp}(t-t_0) \longleftrightarrow H_{lp}(j\omega)\, e^{-j\omega t_0}$$

$$Y(j\omega) = X(j\omega)\, H_{lp}(j\omega)\, e^{-j\omega t_0}$$

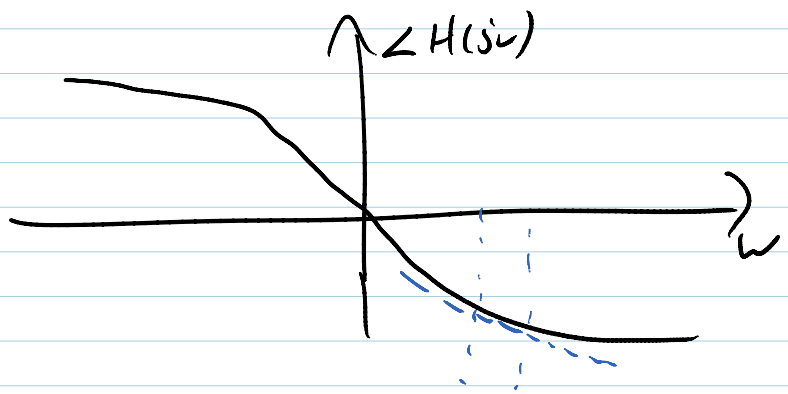
$$y(t) = (x * h_{lp})(t-t_0)$$

$\angle = -\omega t_0 \rightarrow$ linear in ω is okay $\rightarrow$ just a delay in time

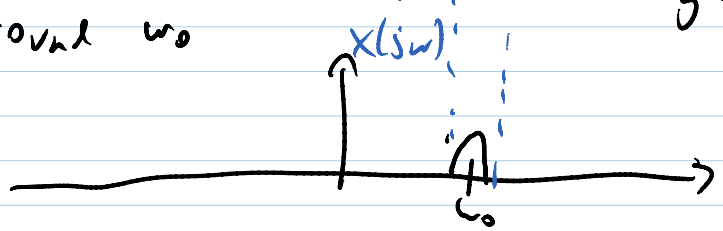
but if $e^{j g(\omega)}$, where $g(\omega)$ nonliner

not as straightforward.

<u>Analysis if $\angle H(j\omega)$ is non-linear</u>



If we have a narrow band signal x centered around ω_0



then the phase is approximately linear

$$\angle H(j\omega) \approx -\phi - \omega\alpha \qquad \text{around } \omega_0$$

$$\alpha = -\frac{d}{d\omega} \angle H(j\omega)\Big|_{\omega=\omega_0}$$

α is called the group delay at ω_0

$$Y(j\omega) \approx X(j\omega)\,|H(j\omega)|\,e^{j(-\phi-\omega\alpha)}$$

$$= X(j\omega)\,|H(j\omega)|\,e^{-j\phi}\,e^{-j\omega\alpha}$$

delay of α in time

Any signal can be thought of as a sum of signals limited to small freq. bands.

$$\alpha(\omega) = \tau(\omega) = -\frac{d}{d\omega} \angle H(j\omega)$$

each little frequency band will be delayed in time by $\tau(\omega)$ (Matlab example here)

Discrete time Filtering

FIR → Finite impulse Response
IIR → Infinite impulse Response

FIR

$$y[n] = a\,x[n] + b\,x[n-1]$$

what is the impulse response?

$$h[n] = a\,\delta[n] + b\,\delta[n-1]$$

$$y[n] = \sum_{k=-N}^{M} b_k\,x[n-k]$$ → FIR

impulse response length is $M+N+1$

IIR

$$y[n] = x[n] + \alpha\,y[n-1] \qquad \alpha < 1$$

$$h[n] = \sum^{\infty} \alpha^k\,\delta[n-k]$$ → infinite duration

$$h[n] = \sum_{k=0}^{\infty} \alpha^k \delta[n-k] \rightarrow \text{infinite duration}$$

In general: Linear Constant coefficient Difference Eqn.

$$\sum_{k=-N_1}^{M_1} a_k y[n-k] = \sum_{\ell=-N_2}^{M_2} b_\ell x[n-\ell]$$

If the initial conditions are 0, then this system is linear. Can "easily" obtain $H(e^{j\omega})$ by taking the FT of this equation

$$\text{FT} \rightarrow \sum_{k=-N_1}^{M_1} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{\ell=-N_2}^{M_2} b_\ell X(e^{j\omega}) e^{-j\omega \ell}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=-N_2}^{M_2} b_k e^{-j\omega k}}{\sum_{\ell=-N_1}^{M_1} a_\ell e^{-j\omega \ell}}$$

Polynomial in $e^{-j\omega}$

So a rational function.

In an FIR filter, $a_k = 0$ for $k \neq 0$, $a_0 = 1$.
we have fewer parameters in an FIR filter and manipulating the numerator of the transfer function doesn't give us as much freedom as in the IIR case where we can also change the denominator, so FIR filters in general require $M_2 + N_2 + 1$ to be big to accomplish the same thing as a smaller order IIR filter.

in general making filters involves choosing a_k, b_ℓ appropriately

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Example Simple FIR filter - Moving average.

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

then $$H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-j\omega k}$$

$$= \frac{1}{N+M+1} e^{j\omega[N-M]/2} \cdot \frac{\sin(\omega(M+N+1)/2)}{\sin(\omega/2)}$$

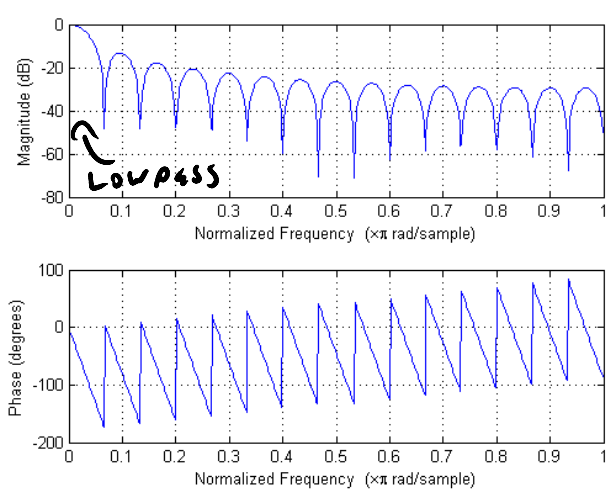
phase $= \omega[N-M]/2$ ← linear

$$\tau(\omega) = -\frac{d}{d\omega}\text{phase}(\omega) = -\frac{(N-M)}{2}$$ ← constant so all freqs are delayed by the same time shift.

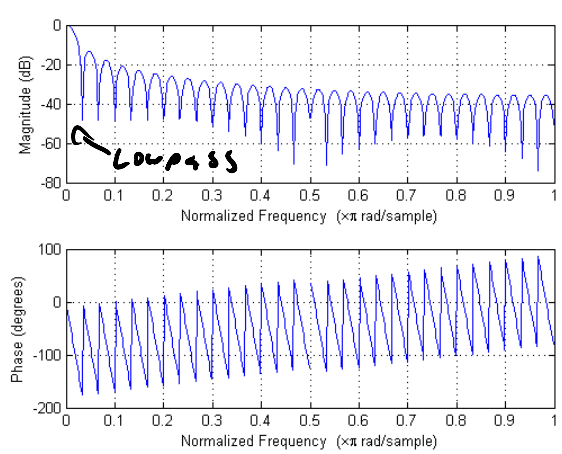
It turns out that all FIR filters have linear phase so $\tau(\omega)$ is constant, so when using FIR filters phase is generally not an issue.

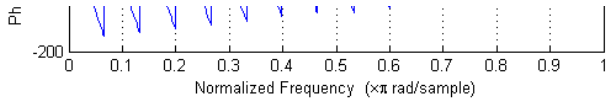
Note: with increased order $M+N+1$ the width of the main lobe of $\frac{\sin(\omega(M+N+1)/2)}{\sin(\omega/2)}$ decreases (it's a sinc-like function)

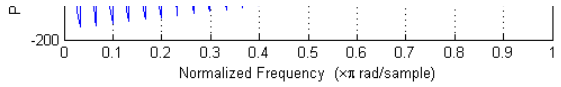
$N=0,\ M=29$



$N=0,\ M=59$







Smaller cutoff freq. as $N+M+1 \nearrow$.