The DFT (half the story)

DFT (Discrete Fourier Transform)

of a finite duration signal (discrete time)

\[ y[n] = \sum_{k=0}^{N-1} x[k] e^{-j\frac{2\pi k n}{N}} \]

Discrete time Signal \( x[n] \)

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

\[ y[n] = X[n] W[n] \]

\[ y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \]

DFT is sampling \( Y(e^{j\omega}) \) at \( \omega = 0, \frac{2\pi}{N}, \frac{2\pi}{N}, ..., \frac{2\pi(N-1)}{N} \)

\[ X[n] = \sin(\omega_0 n) \hat{X}(e^{j\omega}) \]

\[ W(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2} \]
LTI Systems
- Completely characterized by impulse response

**Time Domain**
\[ \begin{align*}
X & \xrightarrow{h} Y \\
Y(t) &= (x * h)(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) \, d\tau
\end{align*} \]

**Freq. Domain**
\[ \begin{align*}
X & \xrightarrow{H} Y \\
Y(j\omega) &= \mathcal{F}\{x(t)\}
\end{align*} \]

\[ Y(j\omega) = X(j\omega) H(j\omega) \]

So \( H \) modulates \( X \), the FT of \( x(t) \)

\( H(j\omega) \) is called the Frequency Response
or Transfer Function of the System

Often useful to look at \( H \) to see how it affects the different freq. components of the input signal.

Let \( z \) be a complex number

\[
z = a + jb, \quad a, b \in \mathbb{R}
\]

\[
|z| = 12\sqrt{\arg(z^2)} = 12|z| = j H(\omega)
\]

\[
H(\omega) = |H(\omega)| e^{j \angle H(\omega)}
\]

So our eqn from above \( Y(\omega) = H(\omega)X(\omega) \)

\[
|Y(\omega)| = |H(\omega)||X(\omega)|
\]

\[
\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)
\]

\[
\begin{align*}
\angle (z_1, z_2) &= \angle z_1 + \angle z_2
\end{align*}
\]

Example \( x(t) = e^{j\omega_0 t} \)

\[
X(\omega) = \frac{2\pi}{j(\omega - \omega_0)}
\]

Then \( Y(\omega) = H(\omega) \left[ \frac{2\pi}{j(\omega - \omega_0)} \right] \)

\[
Y(\omega) = H(\omega_0) \left[ \frac{2\pi}{j(\omega - \omega_0)} \right]
\]

\[
Y(t) = \frac{1}{j}\left( H(\omega_0)2\pi \int (\omega - \omega_0) \right) = H(\omega_0) e^{j\omega_0 t}
\]

\[
= |H(\omega_0)| e^{j\omega_0 t + \angle H(\omega_0)}
\]

In general can look at \( H \) as 2 systems in Cascade

\[
X \rightarrow [H] \rightarrow [e^{j\angle H}]
\]
Filters

A filter is a system that is used to shape the frequency properties of a signal.

Ideal Filters

Lowpass

Highpass

Bandpass

Let's look at the IFT of these filters
Let's look at the IFT of these filters

**Lowpass**

\[ H(\omega) = \begin{cases} 1 & \text{if } |\omega| < \omega_c \\ 0 & \text{o.w.} \end{cases} \]

\[ h(t) = \frac{\sin \omega_c t}{\pi t} = \frac{\omega_c}{\pi} \text{sinc} \left( \frac{\omega_c t}{\pi} \right) \]

- Non-causal
- Infinite duration
- Decays very slowly \( \sim \frac{1}{t} \)

Approximation

\[ |h(t)| = \left| \frac{\sin \omega_c t}{\pi t} \right| \leq \frac{1}{\pi |t|} \sim \frac{1}{|t|} \]

\[ h(t) = \frac{1.5 \omega_c}{\pi} \text{sinc} \left( 1.5 \omega_c t \right) \cdot \frac{\omega_c}{2 \pi} \text{sinc} \left( \frac{\omega_c t}{2 \pi} \right) \]

\[ \text{Dies out } \sim \frac{1}{t^2} \]

\[ |\sin ct| = \left| \frac{\sin \pi t}{\pi t} \right| \leq \frac{1}{\pi t} \]

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\[
\left|\text{Sinc} t\right| = \left|\frac{\sin \pi t}{\pi t}\right| \leq \frac{1}{\pi t}
\]

**Highpass**

\[
H_{\text{HP}}(\omega) = 1 - H_{\text{LP}}(\omega)
\]

\[
h(t) = \delta(t) - \frac{\omega_c}{\pi} \text{sinc} \frac{\omega_c t}{\pi}
\]


\[
h_{\text{BP}}(t) = h(t) e^{j\omega_0 t} + h^*(t) e^{-j\omega_0 t} = 2h_{\text{LP}}(t) \left[ e^{j\omega_0 t} + e^{-j\omega_0 t} \right]
\]

\[
= 2h_{\text{LP}}(t) \cdot \cos(\omega_0 t) = 2\cos(\omega_0 t) \frac{\omega_c}{\pi} \text{sinc} \frac{\omega_c t}{\pi}
\]

Cos in the envelope of \(\text{sinc}\).

**Phase**

\[
1 - e^{-j\omega_0 t}
\]
\[ h_{lp}(t-t_0) \leq H_{lp}(j\omega)e^{-j\omega t_0} \]

\[ Y(j\omega) = X(j\omega) H_{lp}(j\omega)e^{-j\omega t_0} \]

\[ y(t) = (k \ast h_{lp})(t-t_0) \]

\[ \omega = -\omega_0 \] - linear in \( \omega \) is okay \( \rightarrow \) just a delay in time

but if \( e^{jg(\omega)} \) where \( g(\omega) \) nonlinear
not as straightforward.

Analysis if \( \angle H(j\omega) \) is Non-linear

If we have a narrow band signal \( X(j\omega) \) centered around \( \omega_0 \)

then the phase is approximately linear

\[ \angle H(j\omega) \approx -\phi - \omega \delta \] around \( \omega_0 \).
\[ \alpha = -\frac{d}{du} \left| H(j\omega) \right|_{\omega = \omega_0} \]

\( \alpha \) is called the group delay at \( \omega_0 \)

\[ y(j\omega) = x(j\omega) \cdot \left| H(j\omega) \right| e^{j(-\phi - \omega \alpha)} \]

\[ = x(j\omega) \cdot \left| H(j\omega) \right| e^{-\omega \alpha} \]

Any signal can be thought of as a sum of signals limited to small freq. bands.

\[ \alpha(w) = \left| \mathbb{L}(w) \right| = -\frac{d}{du} \left| H(j\omega) \right| \]

Each little frequency band will be delayed in time by \( T(w) \)

**Discrete time Filters**

**FIR** → Finite impulse response

**IIR** → Infinite impulse response

**FIR**

\[ y[n] = a_0 x[n] + b_0 x[n-1] \]

What is the impulse response?

\[ h[n] = a_0 \delta[n] + b_0 \delta[n-1] \]

\[ y[n] = \sum_{k=-N}^{M} b_k x[n-k] \rightarrow \text{FIR} \]

Impulse response length is \( M+N+1 \)

**IIR**

\[ y[n] = x[n] + a \cdot y[n-1] \quad a < 1 \]

\[ h[n] = \sum_{k=0}^{\infty} a^k \delta[n-k] \rightarrow \text{infinite duration} \]
\[ h[n] = \sum_{k=0}^{\infty} a_k x[n-k] \quad \text{infinite duration} \]

In general: Linear constant coefficient Difference Eqn.
\[ \sum_{k=-N_1}^{M_1} a_k y[n-k] = \sum_{l=-N_2}^{M_2} b_l x[n-l] \]

If the initial conditions are 0, then this system is linear. Can easily obtain \( H(e^{j\omega}) \) by taking the FT of this equation.

\[ \text{FT} \rightarrow \sum_{k=-N_1}^{M_1} a_k Y(e^{j\omega}) e^{-j\omega k} = \sum_{l=-N_2}^{M_2} b_l X(e^{j\omega}) e^{-j\omega l} \]

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=-N_2}^{M_2} b_k e^{-j\omega k}}{\sum_{l=-N_1}^{M_1} a_l e^{-j\omega l}} \text{Polynomial in } e^{-j\omega} \text{ So a rational function} \]

In an FIR filter, \( a_k = 0 \) for \( k \neq 0 \), \( a_0 = 1 \). We have fewer parameters in an FIR filter and manipulating the numerator of the transfer function doesn’t give us as much freedom as in the IIR case where we can also change the denominator. So FIR filters in general require \( M_2 + N_2 + 1 \) to be big to accomplish the same thing as a smaller order IIR filter.

In general making filters involves choosing \( a_k \), \( b_l \) appropriately.
Example: Simple FIR filter - Moving average.

\[ y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k] \]

Then \( H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-j\omega k} \)

\[ = \frac{1}{N+M+1} \frac{j\omega [N-M]/2}{\sin((\omega(N+M+1)/2))} \cdot \frac{\sin(\omega/2)}{\sin(\omega/2)} \]

\[ \text{phase} = \frac{\omega [N-M]/2}{\text{linear}} \]

\[ \tau(\omega) = -\frac{d}{d\omega} \text{phase}(\omega) = -\frac{(N-M)}{2} \]

It turns out that all FIR filters have linear phase so \( \tau(\omega) \) is constant, so when using FIR filters phase is generally not an issue.

Note: with increasing order \( M+N+1 \) the width of the main lobe of \( \frac{\sin((\omega(N+M+1)/2))}{\sin(\omega/2)} \) decreases (it's a sinc-like function)

\[ N = 0, \quad M = 29 \]

\[ N = 0, \quad M = 59 \]
Smaller cut-off freq. as $N+1/2$. 