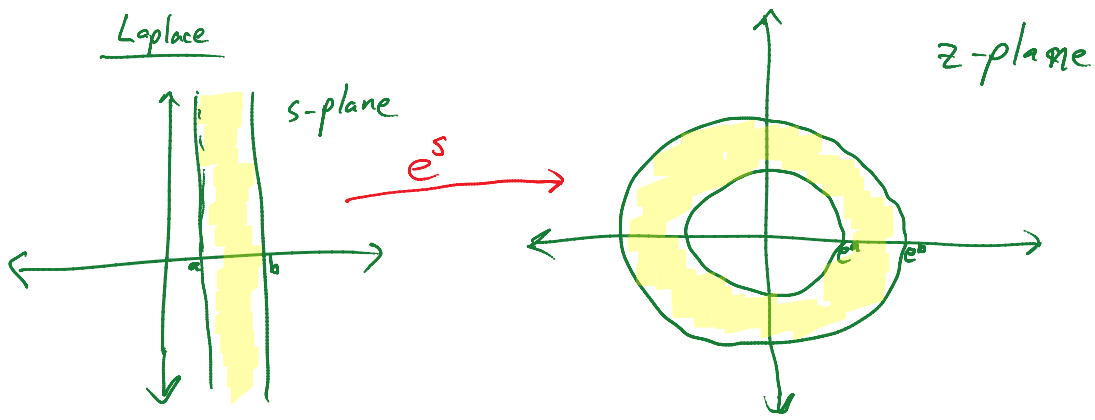
Lecture 18

Tuesday, November 29, 2011
1:21 PM

Z-transform: (Discrete-time)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Looks like Laplace transform with $z = e^s$



Finite-duration:

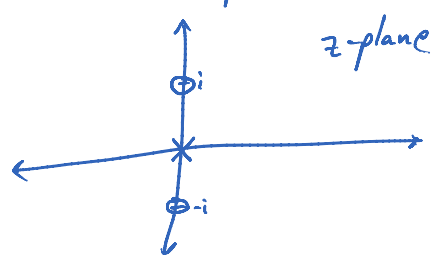
$x[n] = \delta[n+1] + \delta[n-1]$

$\Downarrow$

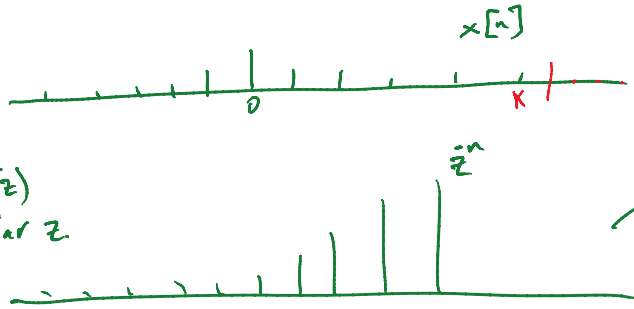
What is the z-transform?

$X(z) = z + z^{-1}$ (rational) $\frac{z^2+1}{z}$

ROC is the entire z-plane with the point $z = 0$.



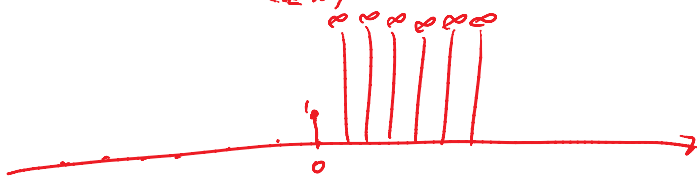
Behavior of z-transform at $z=0$ and $z=\infty$.



Evaluate $X(z)$ at a particular z

Multiply and sum (inner product)

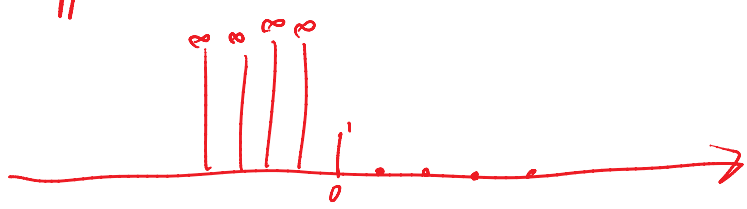
What happens when $z=0$? $\left(\lim_{z\to 0}\right)$



If $x[n]=0 \quad \forall n>0$ then no problem. $X(0)=x[0]$?

If $\exists n>0,\ x[n]\neq 0$ then pole at $z=0$.

What happens when $z\to\infty$



If $x[n]$ is causal, then no poles at $z=\infty$.

(Definition of pole at infty.: $\lim_{z\to\infty} X(z)=\infty$)

Initial Value Th$^{\underline{m}}$: If $x[n]$ is causal: $\lim_{z\to\infty} X(z) = x[0]$

Poles and Zeros at infinity (implied):

Assume $X(z)$ is rational

$$X(z)=\frac{N(z)}{D(z)}$$

Number of zeros is Order of $N(z)$
Number of poles is order of $D(z)$

$X(z) = \frac{N(z)}{D(z)}$ Number of poles is order of $D(z)$

What is $\lim_{z \to \infty} X(z)$?

If $\text{Order}(N) > \text{Order}(D)$, $(\text{Order}(N) - \text{Order}(D))$ <u>Poles</u> at infinity.

If $\text{Order}(D) > \text{Order}(N)$, $(\text{Order}(D) - \text{Order}(N))$ <u>Zeros</u> at infinity.

<u>Bottom Line:</u>

All rational functions have the same number of poles and zeros if you include the ones at infinity.

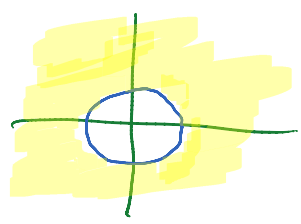
<u>ROC Properties:</u>

1.) ROC is a ring in the z-plane.

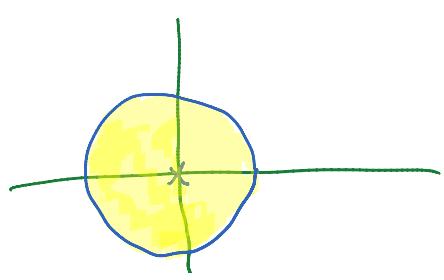
2.) ROC does not contain any poles.

3.) If $x[n]$ is of finite duration, then ROC is the entire z-plane, except perhaps $z=0$ and/or $z=\infty$.

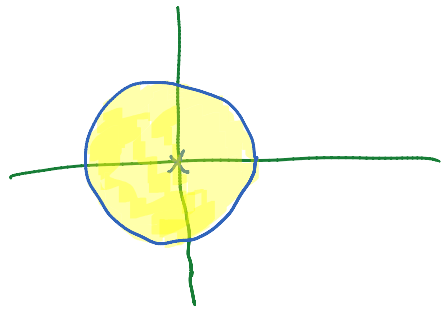
4.) If $x[n]$ is right-sided, then the ROC has no outward bound. (could have poles at infinity).



5.) If $x[n]$ is left-sided, then the ROC has no <u>positive</u> inward bound. (could have poles at zero).



(could have poles at zero).



Assume $X(z)$ is rational:

7.) ROC is bounded by poles.

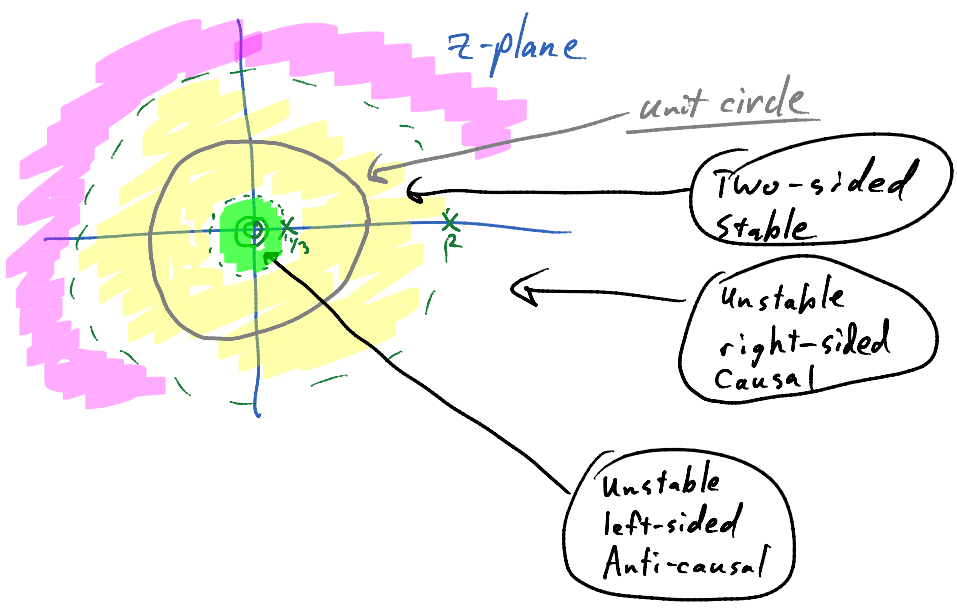
8.) If $x[n]$ is right-sided, then ROC is outside the outermost pole, and causal iff there are more finite poles than finite zeros.

9.) If $x[n]$ is left-sided, then ROC is inside the innermost <u>non-zero</u> pole, and anti-causal iff ROC includes $z=0$.

(Anti-causal: $x[n]=0 \quad \forall n>0$).

Examples:

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} = \frac{z^2}{(z-\frac{1}{3})(z-2)}$$



Inverse:

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Any circle (centered at the origin) in the ROC.

Practical Methods:

Rational: Partial Fraction Expansion

Long-division.

Example:

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} = \frac{-\frac{1}{5}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{6}{5}}{1-2z^{-1}}$$

Consider green ROC from above (i.e. left-sided).

$$x[n] = \underbrace{-\frac{1}{5}\left(-\left(\frac{1}{3}\right)^n u[-n-1]\right)}_{\text{unstable}} + \frac{6}{5}\left(-(2)^n u[-n-1]\right)$$

Long-division:

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} = \frac{1}{1-\frac{7}{3}z^{-1}+\frac{2}{3}z^{-2}}$$

$$\begin{array}{r|l} & 1 + \frac{7}{3}z^{-1} \ldots \\ \hline 1-\frac{7}{3}z^{-1}+\frac{2}{3}z^{-2} & 1 \\ & -\left(1-\frac{7}{3}z^{-1}+\frac{2}{3}z^{-2}\right) \\ \hline & \frac{7}{3}z^{-1}-\frac{2}{3}z^{-2} \\ & -\left(\frac{7}{3}z^{-1} \ldots\right) \end{array}$$

Causal

Oops.

$$\begin{array}{r|l} & \frac{9}{2}z^{2}+\frac{21}{4}z^{3}+\ldots \\ \hline 2\ -2\ 7\ -1\ 1 & \end{array}$$

$$
\begin{array}{r|l}
 & \frac{3}{2}z^2 + \frac{21}{4}z^3 + \cdots \\
\hline
\frac{2}{3}z^{-2} - \frac{7}{3}z^{-1} + 1 & 1 \\
 & -\left(1 - \frac{7}{2}z + \frac{3}{2}z^2\right) \\
\hline
 & \frac{7}{2}z - \frac{3}{2}z^2
\end{array}
$$