Z-transform:
\[
X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}
\]

Looks like Laplace transform with \( z = e^s \)

Finite-duration:
\[
x[n] = \delta[n] + \delta[n-1]
\]

What is the \( Z \)-transform?

ROC is the entire \( Z \)-plane with the point \( z = 0 \).

Behavior of \( Z \)-transform at \( z = 0 \) and \( z = \infty \).
What happens when $z = 0$?

If $x[n] = 0 \; \forall n > 0$ then no problem.

$X(0) = x[0]$.

If $\exists n \geq 0, x[n] \neq 0$ then pole at $z = 0$.

What happens when $z \to \infty$?

If $x[n]$ is causal, then no poles at $z = \infty$.

Definition of pole at infinity: $\lim_{z \to \infty} X(z) = \infty$.

Initial Value Theorem: If $x[n]$ is causal: $\lim_{z \to \infty} X(z) = x[0]$.

Poles and zeros at infinity (implied):

Assume $X(z)$ is rational

\[ X(z) = \frac{N(z)}{D(z)} \]

Number of zeros is order of $N(z)$

Number of poles is order of $D(z)$
\[ X(z) = \frac{D(z)}{D(\zeta)} \]

Number of poles is order of \( D(z) \)

What is \( \lim_{\zeta \to \infty} X(\zeta) \)?

If \( \text{Order}(N) > \text{Order}(D) \), \( (\text{Order}(N) - \text{Order}(D)) \) \textit{poles} at infinity.

If \( \text{Order}(D) > \text{Order}(N) \), \( (\text{Order}(D) - \text{Order}(N)) \) \textit{zeros} at infinity.

**Bottom Line:**

All rational functions have the same number of poles and zeros if you include the ones at infinity.

**ROC Properties:**

1.) ROC is a ring in the \( z \)-plane.

2.) ROC does not contain any poles.

3.) If \( x[n] \) is of finite duration, then ROC is the entire \( z \)-plane, except perhaps \( z = 0 \) and/or \( z = \infty \).

4.) If \( x[n] \) is right-sided, then the ROC has no outward bound (could have poles at infinity).

5.) If \( x[n] \) is left-sided, then the ROC has no positive inward bound (could have poles at zero).
Assume $X(z)$ is rational:

7.) ROC is bounded by poles.

8.) If $x[n]$ is right-sided, then ROC is outside the outermost pole, and causal iff there are more finite poles than finite zeros.

9.) If $x[n]$ is left-sided, then ROC is inside the innermost non-zero pole, and anti-causal iff ROC includes $z=0$.

(Anti-causal: $x[n] = 0 \ \forall n > 0$.)

Examples:

$$X(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})} = \frac{z^2}{(z-\frac{1}{2})(z-2)}$$
Inverse:

\[ x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} \, dz \]

Any circle (centered at the origin) in the ROC.

Practical Methods:

Rational: Partial Fraction Expansion

Long-division.

Example:

\[ X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})} = \frac{-\frac{1}{5}}{1 - \frac{1}{5}z^{-1}} + \frac{\frac{6}{5}}{1 - 2z^{-1}} \]

Consider green ROC from above (i.e. left-sided).

\[ x[n] = \frac{-1}{5} \left( -\left(\frac{1}{3}\right)^n \, u[n-1] \right) + \frac{6}{5} \left( - (2)^n \, u[n-1] \right) \]

unstable

Long-division:

\[ X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - 2z^{-1})} = \frac{1}{1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2}} \]

\[ 1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2} \]

\[ \frac{1}{1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2}} \]

\[ \frac{-1}{\left(1 - \frac{2}{3}z^{-1} + \frac{2}{3}z^{-2}\right)} \]

\[ \frac{\frac{7}{3}z^{-1} - \frac{2}{3}z^{-2}}{-\left(\frac{2}{3}z^{-1} - \cdots \right)} \]

\[ \frac{2}{3}z^{-1} + \frac{21}{4}z^{-2} + \cdots \]

Causal

Oops.
\[ \frac{2}{3} z^2 - \frac{7}{3} z^{-1} + 1 \sqrt{1 - \left(1 - \frac{7}{3} z + \frac{3}{2} z^2 \right) \frac{z}{z^2 - \frac{3}{2} z^2}} \]