Lecture 5
ELE 301: Signals and Systems

Prof. Paul Cuff

Princeton University

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History of the Fourier Series

- Euler (1748): Vibrations of a string
- Fourier: Heat dynamics
- Dirichlet (1829): Convergence of the Fourier Series
- Lagrange: Rejected publication

What is the Fourier Series

- The Fourier Series allows us to represent periodic signals as sums of sinusoids.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi f_0 t}$$

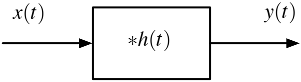
where $f_0 = 1/T_0$ and T_0 is the fundamental period.

- There are other transforms for representing signals
  - Wavelet transform
  - Taylor expansion
  - Any orthonormal basis

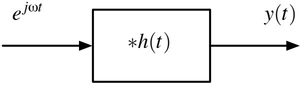
Response of LTI Systems to Exponential Functions

For an LTI system with impulse response $h(t)$, output is the convolution of input and impulse response:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$$



If the input is a complex exponential $x(t) = e^{j\omega t}$



Eigenfunctions

Continuous time:

$$e^{st} \longrightarrow^{h} H(s)e^{st}$$

Discrete time:

$$z^n \longrightarrow^{h} H(z)z^n$$

Aliasing

Wolfram Demo:

$$e^{(\sigma+j(2\pi f))n} = e^{(\sigma+j(2\pi(f+k)))n} \text{ for all integers } n \text{ and } k.$$

Sums of Exponentials

$$a_1 + e^{s_1 t} + a_2 + e^{s_2 t} + a_3 + e^{s_3 t} \longrightarrow^h a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Period Signals

Claim:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi f_0 t}$$

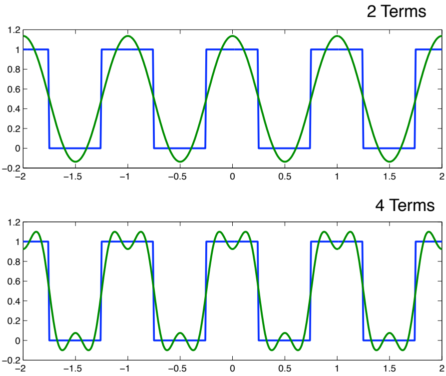
where $f_0 = 1/T_0$ and T_0 is the fundamental period.

Consider an easy one:

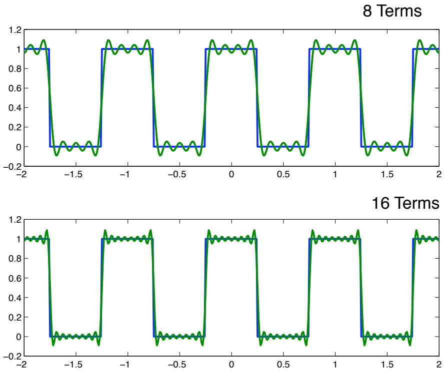
$$\begin{aligned} x(t) &= \cos(2\pi f_0 t) \\ &= \frac{1}{2} e^{2\pi f_0 t} + \frac{1}{2} e^{-2\pi f_0 t}. \end{aligned}$$

Therefore, $T = 1/f_0$ and $a_1 = a_{-1} = 1/2$.

Fourier Series approximation to a square wave



Fourier Series approximation to a square wave



Real Signals

If x is real

$$x(t) \quad = \quad a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k2\pi f_0 t + \theta_k),$$

where $A_k e^{j\theta_k} = a_k$.

Fourier Series Coefficients

$$a_k \quad = \quad \frac{1}{T}\int_T x(t)e^{-jk2\pi f_0 t}dt.$$

Conditions for Convergence

- Continuous
- Finite Power (energy over a period)
- Dirichlet conditions:
  - Absolutely integrable
  - Bounded Variation
  - Finite Discontinuities

Linearity

Time-shift

Time Reversal

Time Scaling

Multiplication

Conjugate

Parseval's Theorem

Discrete Time

Aliasing:

All periodic exponential signals with period N are:

$$\phi_k[n] \;=\; e^{jk\frac{2\pi}{N}n} \text{ for } k = 0, 1, ..., N-1.$$

Discrete Time Fourier Series

$$\begin{aligned} x[n] &= \sum_{k=<N>} a_k \phi_k[n] \\ a_k &= \frac{1}{N} \sum_{n=<N>} x[n]\phi_k[-n] \end{aligned}$$

Multiplication

Fourier Series Example

Fourier Series Example using Matlab

$$x(t) = e^{-t} \quad \text{for} \quad -1 < t \leq 1.$$