

Lecture 10

Thursday, October 16, 2014
10:26 AM

Coherent Communication (Harrow, 2004)

$$[\overline{q} \rightarrow \overline{q}\overline{q}] \\ \text{Isometry : } |i\rangle^A \xrightarrow{\text{?}} |i\rangle^A |i\rangle^B \Rightarrow U = \sum_i |i\rangle^A |i\rangle^B \langle i|$$

Any orthonormal basis

$$N_u^{A \rightarrow AB}(\rho) = \sum_{ij} |i\rangle^A |i\rangle^B \langle i| \rho |j\rangle \langle j|$$

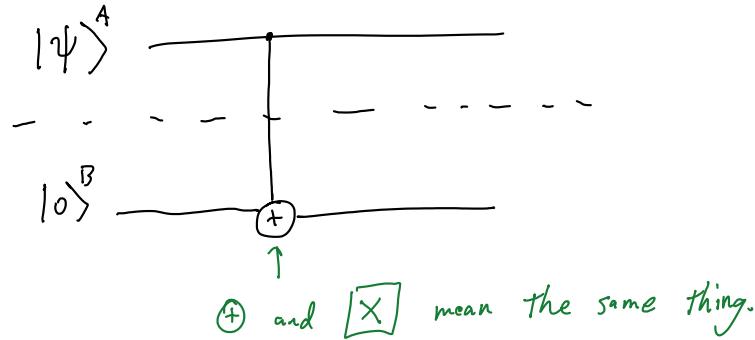
$$\Rightarrow \text{Tr}_A(N_u(\rho)) = \text{Tr}_A\left(\sum_{ij} \langle i| \rho |j\rangle |i\rangle \langle j|^A \otimes |i\rangle \langle j|^B\right)$$

$$= \sum_i \langle i| \rho |i\rangle |i\rangle \langle i| = [c \rightarrow c]$$

Measure with $\{|i\rangle\}$
 and forget measurement
 $[\overline{q} \rightarrow \overline{q}\overline{q}]$ retains coherent measurement

\Rightarrow Coherent Bit Channel is isometric extension of $[c \rightarrow c]$
 where the sending system retains the environment.

Consider Coherent Bit Channel as remote C-NOT ("remote controlled" NOT gate)
 (in computational basis)

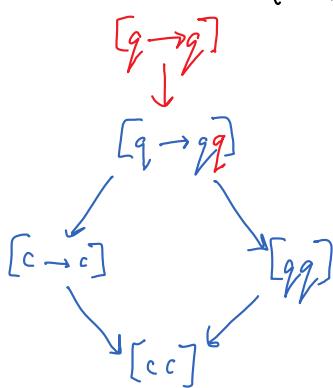


- Inequalities :
- 1.) $[\overline{q} \rightarrow \overline{q}] \geq [\overline{q} \rightarrow \overline{q}\overline{q}]$
 Proof : Alice locally performs C-NOT on $|\psi\rangle^A |0\rangle^B$
 Alice sends B' to Bob.
 - 2.) $[\overline{q} \rightarrow \overline{q}\overline{q}] \geq [c \rightarrow c]$
 Proof : Ignore ("trace out") A after operation.
 - 3.) $\Gamma_{n \rightarrow m} \geq \Gamma_{0 \rightarrow 1}$

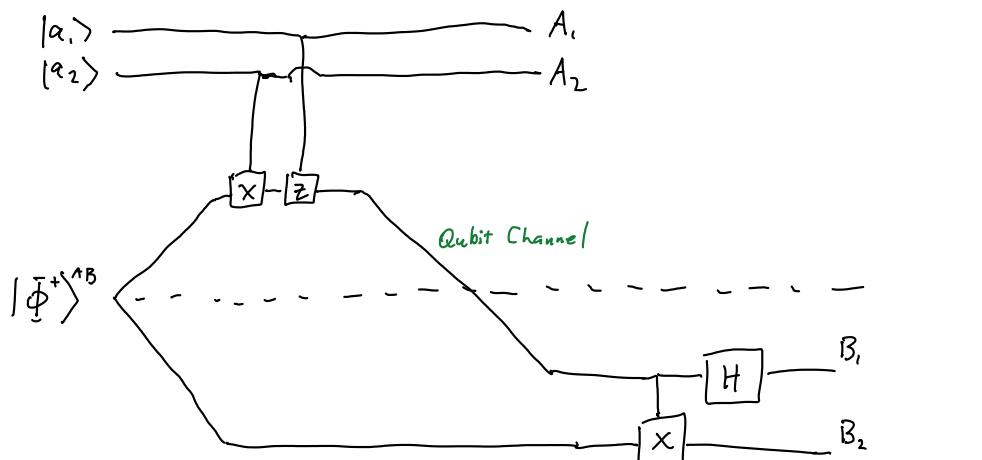
$\therefore \text{I } \mathcal{U} = \mathcal{U}_A - \mathcal{U}_B$

Proof: Alice prepares superposition $|\psi\rangle^A = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$
 Alice uses $[g \rightarrow gg]$

$$U|\psi\rangle = \left(\sum_i |i\rangle^A |i\rangle^B \langle i| \right) |\psi\rangle^A = \frac{1}{\sqrt{2}}(|00\rangle^{AB} + |11\rangle^{AB}) = |\Phi^+\rangle^{AB}$$



Coherent Super-dense Coding:



We know this works because:

$ 00\rangle^{A_1 A_2}$	\longrightarrow	$ 0000\rangle^{A_1 A_2 B_1 B_2}$
$ 01\rangle^{A_1 A_2}$	\longrightarrow	$ 0101\rangle^{A_1 A_2 B_1 B_2}$
$ 10\rangle^{A_1 A_2}$	\longrightarrow	$ 1010\rangle^{A_1 A_2 B_1 B_2}$
$ 11\rangle^{A_1 A_2}$	\longrightarrow	$ 1111\rangle^{A_1 A_2 B_1 B_2}$

$$\Rightarrow \text{Isometry: } \left(\sum_i |i\rangle^A |i\rangle^B \langle i| \right) \otimes \left(\sum_j |j\rangle^B |j\rangle^A \langle j| \right)$$

$[g \rightarrow gg]$ $[g \rightarrow gg]$

$$\Rightarrow [g \rightarrow g] + [g \rightarrow gg] \geq 2 [g \rightarrow gg]$$

Notice, all steps are reversible. Only isometric steps involved (no measurement)

Coherent Teleportation:

Use two coherent bit channels rather than $2[c \rightarrow c]$.

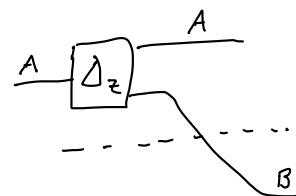
$$(2[g \rightarrow gg] + [gg] \geq [g \rightarrow g])$$

↑ trivial

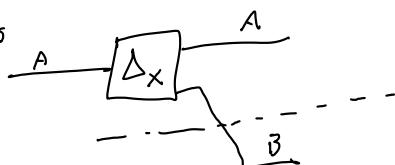
Save entanglement.

Remember: $[g \rightarrow gg]$ can function in any basis.

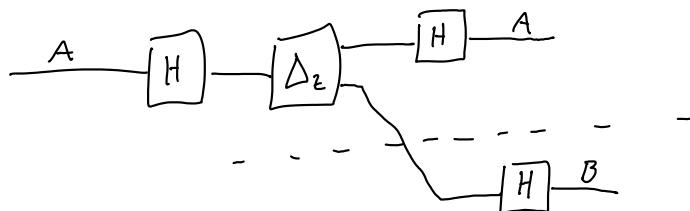
Denote $[g \rightarrow gg]$ in comp. basis as



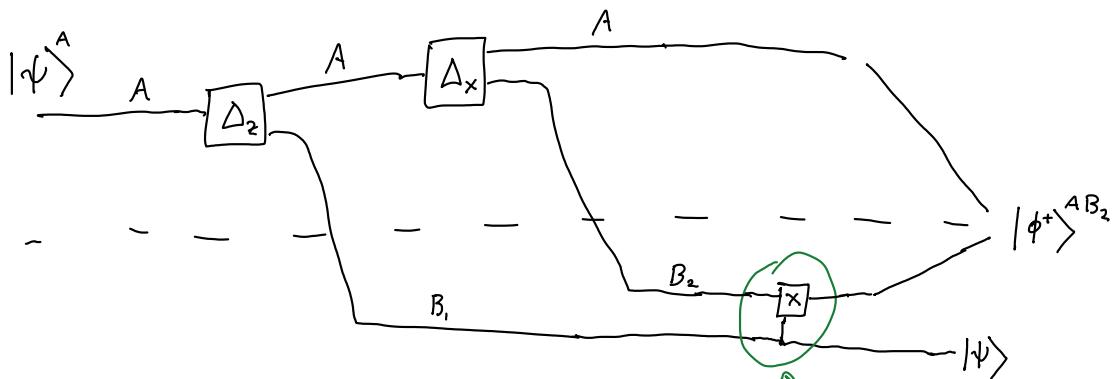
Denote $[g \rightarrow gg]$ in Hadamard basis as
 $\{|\psi\rangle, |\phi\rangle\}$



Easy to construct Δ_x from Δ_z :



Teleportation:



$$2[g \rightarrow gg] \geq [g \rightarrow g] + [gg]$$

Surprising that this works.

$$\text{Analysis: } |\psi\rangle = \alpha|0\rangle^A + \beta|1\rangle^A$$

$$\text{After } \boxed{A_2}: \alpha|0\rangle^A|0\rangle^{B_1} + \beta|1\rangle^A|1\rangle^{B_1} = \alpha\left(\frac{|+\rangle^A + |-\rangle^A}{\sqrt{2}}\right)|0\rangle^{B_1} + \beta\left(\frac{|+\rangle^A - |-\rangle^A}{\sqrt{2}}\right)|1\rangle^{B_1}$$

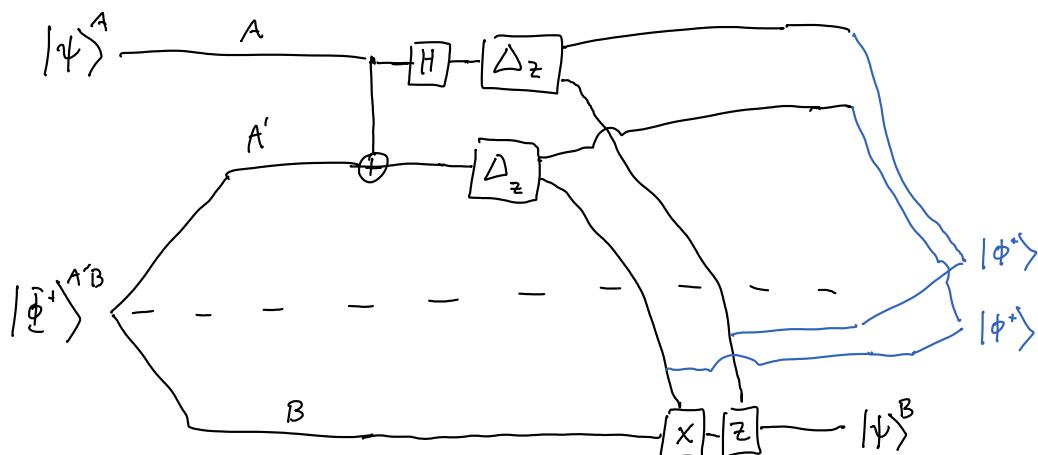
$$\text{After } \boxed{A_X}: \alpha\left(\frac{|+\rangle^A|+\rangle^{B_2} + |-\rangle|-\rangle}{\sqrt{2}}\right)|0\rangle^{B_1} + \beta\left(\frac{|+\rangle^A|+\rangle^{B_2} - |-\rangle|-\rangle}{\sqrt{2}}\right)|1\rangle^{B_1}$$

Notice: C-NOT:

$$\begin{aligned}|0\rangle|+\rangle &\rightarrow |0\rangle|+\rangle \\|0\rangle|-\rangle &\rightarrow |0\rangle|-\rangle \\|1\rangle|+\rangle &\rightarrow |1\rangle|+\rangle \\|1\rangle|-\rangle &\rightarrow -|1\rangle|-\rangle\end{aligned}$$

$$\begin{aligned}\text{After C-NOT: } \alpha\left(\frac{|+\rangle|+\rangle^{B_2} + |-\rangle|-\rangle}{\sqrt{2}}\right)|0\rangle^{B_1} + \beta\left(\frac{|+\rangle|+\rangle + |-\rangle|-\rangle}{\sqrt{2}}\right)|1\rangle \\= |\phi^+\rangle^{AB_2} \otimes |\psi\rangle^{B_1} \quad \square\end{aligned}$$

Another approach to teleportation:



$$2[\bar{g} \rightarrow g\bar{g}] + [g\bar{g}] \geq [\bar{g} \rightarrow \bar{g}] + 2[g\bar{g}]$$

catalyst.

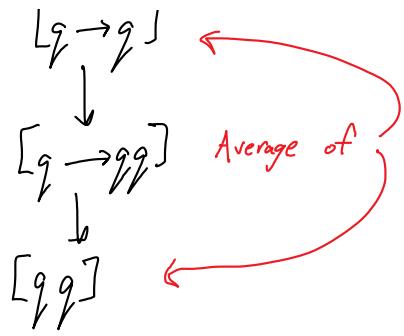
Same inequality.

Resource Equality:

$$2[\bar{g} \rightarrow g\bar{g}] = [\bar{g} \rightarrow \bar{g}] + [g\bar{g}]$$

$$[\bar{g} \rightarrow \bar{g}] \quad \swarrow$$

$$2[g \rightarrow gg] = [g \rightarrow g] + [gg]$$



Information Theory implication:

Entanglement-assisted classical capacity identifies E and C:

$$\langle N \rangle + E[gg] \geq C[c \rightarrow c] \quad \text{Relationship}$$

$$\text{We already claimed that } \langle N \rangle + \left(E + \frac{C}{2}\right)[gg] \geq \frac{C}{2}[g \rightarrow g]$$

In fact, classical comm. can be upgraded to coherent comm.

$$\langle N \rangle + E[gg] \geq C[g \rightarrow gg]$$

$$\Rightarrow \langle N \rangle + \left(E - \frac{C}{2}\right)[gg] \geq \frac{C}{2}[g \rightarrow g]$$

\uparrow
Save entanglement.

There are cases where this produces optimal/ tradeoff
for EA quantum comm.

Overview of Ch. 8 :

Noiseless Resource Capacity Region:

$$\text{Consider } C_u = \{(C, Q, E) : C[c \rightarrow c] + Q[g \rightarrow g] + E[gg] \leq 0\}$$

\uparrow
Unit resource

\uparrow
All feasible resource exchanges,

$$\text{We know : } (-2, 1, -1) \in C_u$$

Teleportation

$$(2, -1, -1) \in C_u$$

Super-dense Coding

$$(0, -1, 1) \in C_u \quad \text{Entanglement distribution}$$

Also, if $v \in C_u$ then we consider $\alpha v \in C_u$ for $\alpha > 0$.

$\Rightarrow C_u$ is a cone.

Also, C_u is convex. Sums are feasible by a sequence of exchanges.

$\Rightarrow C_u$ is a convex cone.

Theorem: C_u is the convex cone generated by:

$$(-2, 1, -1), (2, -1, -1), (0, -1, 1).$$

\uparrow \nearrow \searrow
Extreme rays.

Sanity check:

We should be able to throw away resources.

\Rightarrow If $v \in \mathbb{R}^3_+$ then $v \in C_u$.

Check that this is true for the cone above.

Proof requires a converse:

This cone has three faces. Identify faces:

Cone is generated by $\begin{bmatrix} -2 & 2 & 0 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ where $\alpha, \beta, \gamma \geq 0$

↙ Matrix inverse

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ q \\ e \end{bmatrix}$$

↙ Non-negative

$$\Rightarrow \text{Faces of cone: } \begin{aligned} -\frac{1}{2}c - \frac{1}{2}q - \frac{1}{2}e &\geq 0 \\ -\frac{1}{2}q - \frac{1}{2}e &\geq 0 \\ -\frac{1}{2}c - q &\geq 0 \end{aligned} \Rightarrow \begin{aligned} c + q + e &\leq 0 \\ q + e &\leq 0 \\ c + 2q &\leq 0 \end{aligned}$$

Must argue necessity.

Proof breaks into many cases (see text)

Underlying arguments:

- 1.) Entanglement cannot produce comm.
- 2.) Classical comm. cannot produce quantum resources
- 3.) Holevo bound limit classical comm. from quant. comm.