Lecture 12

Wednesday, October 22, 2014
11:45 AM

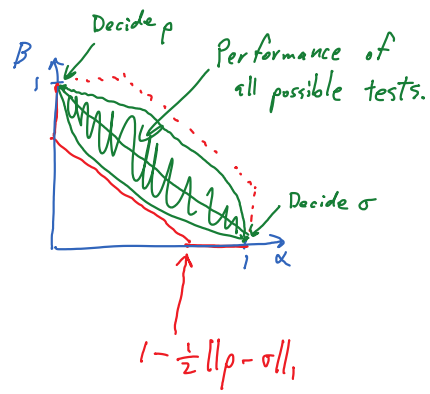
Ch. 9 Continued

Trace distance: $\|\rho-\sigma\|_1 = \mathrm{Tr}\,|\rho-\sigma|$

Relationship to hypothesis testing:

Type 1 error: $\alpha = P(\text{test decides } \sigma \mid \rho)$

Type 2 error: $\beta = P(\text{test decides } \rho \mid \sigma)$



Fidelity:

$$F(|\psi\rangle, |\varphi\rangle) = |\langle\psi|\varphi\rangle|^2$$

$$F(|\psi\rangle, \sigma) = \langle\psi|\rho|\psi\rangle = E_x |\langle\psi|\varphi_x\rangle|^2 = \mathrm{Tr}(\rho\sigma) = \mathrm{Tr}(\sqrt{\sigma}\rho\sqrt{\sigma})$$

$$F(\rho, \sigma) = \max_{|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA}} F(|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA})$$

Uhlmann Fidelity

equal because rank 1

Uhlmann Theorem (Thm 9.2.4)

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}\right)^2$$

Properties:

1.) $0 \le F \le 1$

2.) $F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1, \sigma_1)\, F(\rho_2, \sigma_2)$

3.) Joint Concavity:

$$F\left(\sum_x p_x(x)\rho_x, \sum_x p_x(x)\sigma_x\right) \ge \sum_x p_x(x) F(\rho_x, \sigma_x)$$

$$\Downarrow$$

$$F(\lambda\rho_1 + (1-\lambda)\rho_2, \sigma) \ge \lambda F(\rho_1, \sigma) + (1-\lambda) F(\rho_2, \sigma) \quad \forall \lambda \in [0,1]$$

4.) Monotonicity:

$$F(\rho^{AB}, \sigma^{AB}) \le F(\rho^A, \sigma^A)$$

(Def. involves max.)

5.) Isometric invariance

6.) Monotonicity of noisy channels:

$$F(\rho, \sigma) \le F(\mathcal{N}(\rho), \mathcal{N}(\sigma))$$

Also properties of trace distance.

Connection to probability:

Trace distance: $\|\rho-\sigma\|_1 = \max_{\{\Lambda_x\}} \|p-q\|_1$ $\qquad p(x) = \mathrm{Tr}(\Lambda_x \rho)$, $q(x) = \mathrm{Tr}(\Lambda_x \sigma)$

Fidelity: $F(\rho,\sigma) = \min_{\{\Lambda_x\}} d_B(p,q)$

Bhattacharya coefficient: $d_B(p,q) = \left(\sum_x \sqrt{p(x)q(x)}\right)^2$

Connection between Fidelity and Trace distance:

$$1-\sqrt{F(\rho,\sigma)} \le \tfrac{1}{2}\|\rho-\sigma\|_1 \le \sqrt{1-F(\rho,\sigma)}$$

This is the square of the Hellinger distance.

Uses relationship between total variation and Bhattacharya coefficient.

Equal for pure states.

Gentle Measurement:

Let $0 \le \Lambda \le I$ be a measurement operator. (part of a POVM)

Consider the post-measurement state $\rho' = \dfrac{\sqrt{\Lambda}\rho\sqrt{\Lambda}}{\mathrm{Tr}(\Lambda\rho)}$

$$\|\rho-\rho'\|_1 \le 2\sqrt{1-\mathrm{Tr}(\Lambda\rho)}$$

If you are certain about the measurement outcome, then you don't affect the state much.

Proof: Use fidelity.

Must show that $F(\rho,\rho') \ge \mathrm{Tr}(\Lambda\rho)$

Start with pure state.

Finish by purifying ρ.

For pure state: $\rho = |\psi\rangle\langle\psi|$, $\quad \rho' = \dfrac{\sqrt{\Lambda}|\psi\rangle\langle\psi|\sqrt{\Lambda}}{\langle\psi|\Lambda|\psi\rangle}$

$$F(\rho,\rho') = \frac{\left(\langle\psi|\sqrt{\Lambda}|\psi\rangle\right)^2}{\langle\psi|\Lambda|\psi\rangle}$$

$$\geq \frac{(\langle\psi|\Lambda|\psi\rangle)^2}{\langle\psi|\Lambda|\psi\rangle}$$

$$= \langle\psi|\Lambda|\psi\rangle = \mathrm{Tr}(\Lambda\rho)$$

Gentle Operator:

Slight modification of above.

$$\|\rho - \sqrt{\Lambda}\rho\sqrt{\Lambda}\|_1 \leq 2\sqrt{1-\mathrm{Tr}(\Lambda\rho)}$$

Not a density operator.
Not normalized

Proof uses triangle inequality and Cauchy-Schwarz. No use of fidelity.

Ensembles:

$$\sum_x p(x)\|\rho_x - \sqrt{\Lambda}\rho_x\sqrt{\Lambda}\|_1 \leq 2\sqrt{1-\underbrace{\sum_x p(x)\,\mathrm{Tr}(\Lambda\rho_x)}_{\text{Probability of measurement for ensemble.}}}$$

Average affect on states.

Proof simply uses Jensen's inequality.

Fidelity of a channel:

(Minimum) $F_{min}(\mathcal{N}) = \min_{|\psi\rangle} F(|\psi\rangle, \mathcal{N}(|\psi\rangle\langle\psi|))$ Hard to compute.

Don't need to consider mixed states because of concavity.

(Expected) $\bar{F}(\mathcal{N}) = E_X\, F(X, \mathcal{N}(X))$ where X is a pure state selected uniformly at random.

(Entanglement) $F_e(\rho, \mathcal{N}) = F\left(|\varphi_\rho\rangle^{RA}, (I^R \otimes \mathcal{N}^A)(|\varphi_\rho\rangle^{RA})\right)$

This is stricter than $F(\rho, \mathcal{N}(\rho))$, which only cares about marginals.

That is, $F_e(\rho, \mathcal{N}) \leq F(\rho, \mathcal{N}(\rho))$.

Explicit formula for entanglement fidelity in terms of Kraus operators.

Interesting equality: $\overline{F}(\mathcal{N}) = \dfrac{d\, F_e(\pi, \mathcal{N}) + 1}{d+1}$

Classical Information Measures: Ch. 10

Entropy: Let X be a discrete random variable $\sim p_X$

$$H(X) = E \log \frac{1}{p_X(x)} = \sum_x p_X(x) \log \frac{1}{p_X(x)}$$

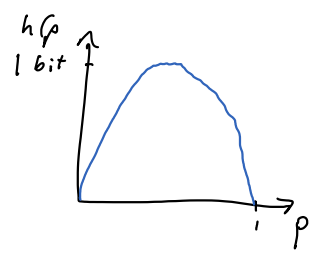
Appears twice

If $\log_2$ then units are bits.

Let $0 \log 0 = 0$.

Example: Binary $X \sim \mathrm{Bern}(p)$

$$H(X) = p \log \frac{1}{p} + (1-p) \log \frac{1}{1-p} = h(p)$$



Property: $0 \leq H(X) \leq \log |\mathcal{X}|$

Proof: 1.) $\log \frac{1}{p_X(x)} \geq 0 \quad \forall x \Rightarrow H(X) \geq 0$

Equality iff $\log \frac{1}{p_X(x)} = 0 \quad \forall x$ s.t. $p_X(x) > 0$

$\Downarrow$

$p_X(x) = 1$

$\Rightarrow$ $\boxed{\text{Deterministic}}$

2.) $H(X) = E \log \frac{1}{p(x)}$

$$\leq \log E \frac{1}{p(X)} \qquad \text{Jensen's inequality}$$

$$= \log \sum_{x \in \mathcal{X}} \frac{p(x)}{p(x)}$$

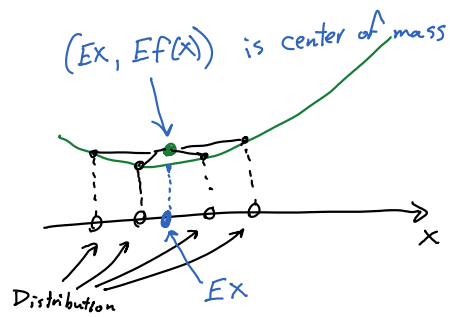
$$= \log |\mathcal{X}|$$

$\boxed{\text{Equality iff uniform}}$ because $\log$ is strictly concave $\Rightarrow \frac{1}{p(x)}$ must be const.

Jensen's Inequality:

If f is a convex function,

$$E f(X) \geq f(EX)$$



Furthermore, if f is strictly convex, then inequality is strict or X is deterministic.