Coordination via Communication

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Stanford University

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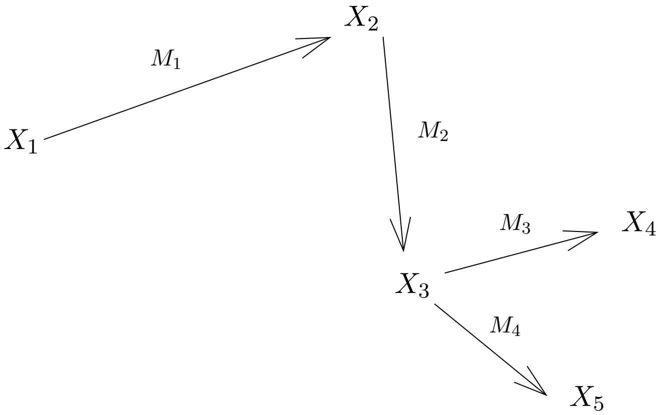
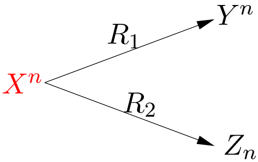
Coordinated Action in a Network



Example: Task Assignment

Computation Tasks: $T = \{0, 1, 2\}$
Processors: X, Y, and Z

$X^n \sim iid, \;\; Unif(\{0, 1, 2\})$



What are the required rates R_1 and R_2?

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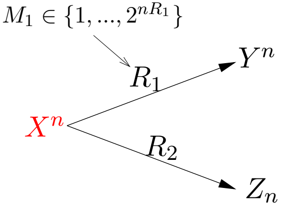

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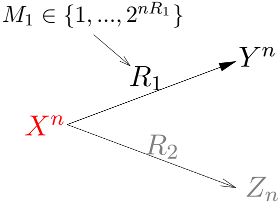

Achievable Rates:

Describe X^n:
$|M_1| = 3^n$

Use codebook of Y^n sequences:
$|M_1| \approx \left(\frac{3}{2}\right)^n$

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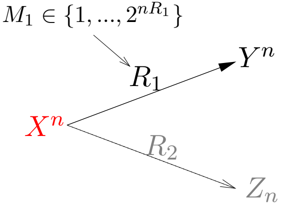

Achievable Rates:

Using rate-distortion-like notation:

$R_1 \geq I(X;Y) = \log_2(3/2)$,
where

$$p(y|x) = \begin{cases} \frac{1}{2}, & y \neq x \\ 0, & y = x \end{cases}$$

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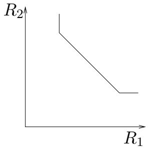

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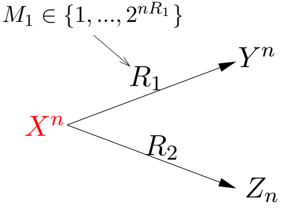

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where

$$p(y|x) = \begin{cases} \frac{1}{2}, & y \neq x \\ 0, & y = x \end{cases}$$

However,
$R_2 \geq I(X,Y;Z) = \log_2 3$

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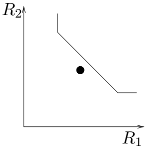
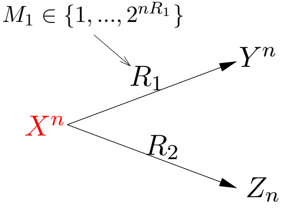


Achievable Rates:

Restrict the support of Y and Z:

$$\begin{aligned} Y &\in \{0,1\}, \\ Z &\in \{1,2\}. \end{aligned}$$



$$\begin{aligned} R_1 &\geq I(X;Y) = \log_2 3 - 2/3, \\ R_2 &\geq I(X;Z) = \log_2 3 - 2/3. \end{aligned}$$

Example: Task Assignment

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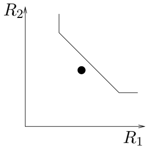
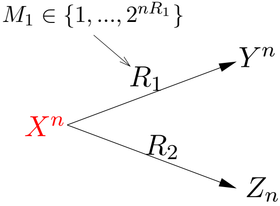


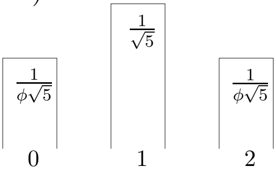
Achievable Rates:



First send $\hat{X}$ to both,
where $\hat{X} = X$ with probability $\frac{1}{\sqrt{5}}$.

$R_{\hat{x}} = 0.04$ bits.

$p(\hat{x}|x = 1)$



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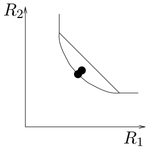

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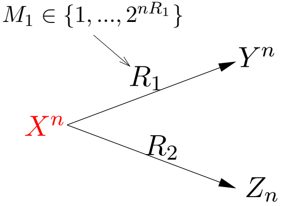


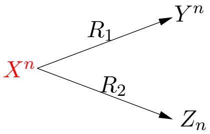
First send $\hat{X}$ to both,
where $\hat{X} = X$ with probability $\frac{1}{\sqrt{5}}$.

$R_{\hat{x}} = 0.04$ bits.

$$\begin{aligned} Y &\in \{\hat{X} - 1, \hat{X}\}, \\ Z &\in \{\hat{X} + 1, \hat{X}\}. \end{aligned}$$

$$\begin{aligned} R_1 &\geq I(X; \hat{X}, Y) = \log_2 3 - \log_2 \phi, \\ R_2 &\geq I(X; \hat{X}, Z) = \log_2 3 - \log_2 \phi. \end{aligned}$$

Broadcast Source Coding



$X \sim iid, p_0(x)$.
Desired Coordingation: $p_0(y, z|x)$.

Histogram:

$$H_{X^n,Y^n,Z^n}(a,b,c) \triangleq \sum_{i=1}^{n} \mathbf{1}(X_i = a, Y_i = b, Z_i = c).$$

Requirement:

$$E \left\| H_{X^n,Y^n,Z^n}(x,y,z) - p_0(x,y,z) \right\|_{TV} \rightarrow 0.$$

Achievable Rates:

$$\begin{aligned} R_1 &\geq I(X;Y,U), \\ R_2 &\geq I(X;Z,U), \\ R_1 + R_2 &\geq I(X;Y,U) + I(X;Z,U) + I(Y;Z|X,U). \end{aligned}$$

for some U.
[Zhang, Berger 95], [Cover, Permuter 07]

Relay Source Coding

$X, Y \sim iid, p_0(x, y)$.
Desired Coordingation: $p_0(z|x, y)$.

$$X^n \xrightarrow{R_1} Y^n \xrightarrow{R_2} Z_n$$

Achievable Rates:

$$\begin{aligned} R_1 &\geq I(X; U, V|Y), \\ R_2 &\geq I(X; U) + I(Y, V; Z|U). \end{aligned}$$

for some U and V satisfying the Markovity,

$$\begin{array}{ccccc} Y & - & X & - & U, V, \\ X & - & U, V, Y & - & Z. \end{array}$$

A Stronger Form of Coordination

Desired Distribution: $p_0(x, y, z)$

Jointly Typical Coordination

$$E \left\| H_{X^n,Y^n,Z^n}(x, y, z) - p_0(x, y, z) \right\|_{TV} \quad \rightarrow \quad 0.$$

Jointly Random Coordination

$$\left\| p(x^n, y^n, z^n) - \prod_{i=1}^{n} p_0(x_i, y_i, z_i) \right\|_{TV} \quad \rightarrow \quad 0.$$

Two Nodes

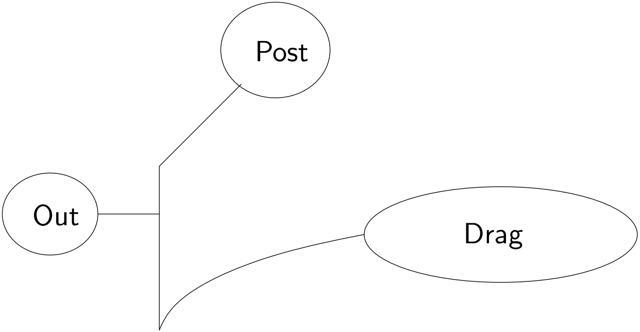
$$X^n \xrightarrow{\quad R_1 \quad} Y^n$$

Jointly Random Coordination (for $p_0(x, y)$):

$$R \geq C(X;Y) = \min_{X-U-Y} I(X,Y;U).$$

[Wyner 75], [Cuff 08]

Football Receiver Routes



Zero-Sum Repeated Game

Payoff Matrix: Π

X	Y	Defense					
A	A	0	0	0	0	0	0
B	B	0	0	0	0	0	0
C	C	0	0	0	0	0	0
A	B	0	6	6	6	6	6
A	C	6	0	6	6	6	6
B	A	6	6	0	6	6	6
B	C	6	6	6	0	6	6
C	B	6	6	6	6	0	6
C	C	6	6	6	6	6	0

Game Value = 5

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B	C	6	6	6	0	6	6
C	B	6	6	6	6	0	6
C	C	6	6	6	6	6	0

Game Value = 5

$$p^*(x,y) = \begin{cases} \frac{1}{6}, & x \neq y \\ 0, & x = y \end{cases}$$

$$R \geq C(X;Y) = \log_2 3.$$

Zero-Sum Repeated Game

Payoff for ith iteration: $\Pi(x_i, y_i, z_i)$

Assume opponent plays optimally.
Expected payoff: $\Theta_i = \min_z E\left[\Pi(X_i, Y_i, z)|X^{i-1}, Y^{i-1}\right]$.

Message:

$$\begin{aligned} M &\in \{1, ..., 2^{nR}\} \\ p(x^n, m, y^n) &= p(m) \prod_{i=1}^{n} p(x_i|m, x^{i-1}, y^{i-1}) p(y_i|m, x^{i-1}, y^{i-1}) \end{aligned}$$

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i.e.

$$X_i - M - Y_i \ \mid \ X^{i-1}, Y^{i-1}$$

Zero-Sum Repeated Game Converse

$$X_i - M - Y_i \ \mid \ X^{i-1}, Y^{i-1}$$
$$\Theta_i = \min_z E\left[\Pi(X_i, Y_i, z) | X^{i-1}, Y^{i-1}\right]$$

Suppose for some n,

$$E\frac{1}{n}\sum_{i=1}^{n}\Theta_i \ \geq \ \Psi.$$

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$$\begin{aligned} nR &\geq H(M) \\ &\geq I(X^n, Y^n; M) \\ &= \sum_{i=1}^{n} I(X_i, Y_i; M | X^{i-1}, Y^{i-1}) \\ &= nI(X_K, Y_K; M | X^{K-1}, Y^{K-1}, K), \end{aligned}$$

where $K \sim Unif(\{1, ..., n\})$.

Zero-Sum Repeated Game Converse

$$X_i - M - Y_i \ | \ X^{i-1}, Y^{i-1}$$
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where $K \sim Unif(\{1, ..., n\})$.
Let $W \triangleq \{X^{K-1}, Y^{K-1}, K\}$.

Zero-Sum Repeated Game Converse

To achieve an average score Ψ.

$$R \;\; > \;\; I(X,Y;M|W)$$

for some X, Y, M, and W that satisfy,

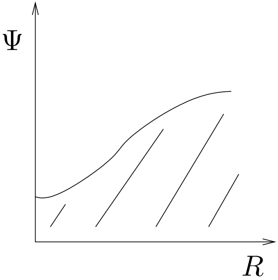
$$X - M - Y|W,$$

and

$$E \min_z E\left[\Pi(X,Y,z)|W\right] \geq \Psi.$$

Zero-Sum Repeated Game Converse

$$
\begin{aligned}
\mathcal{R}_0 \quad \triangleq \quad & \{(R, \Psi) : \exists X, Y \text{ such that} \\
& \quad R \geq C(X;Y) \\
& \quad \min_z E\Pi(X, Y, z) \geq \Psi\}.
\end{aligned}
$$



Zero-Sum Repeated Game Converse

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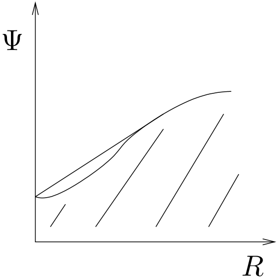

Summary

- Jointly Typical Coordination

$$\begin{tikzcd} & Y^n \\ X^n \arrow[ur, "R_1"] \arrow[dr, "R_2"'] & \\ & Z_n \end{tikzcd}$$

$$X^n \xrightarrow{R_1} Y^n \xrightarrow{R_2} Z_n$$

- Jointly Random Coordination
  - $R \geq C(X;Y)$
  - Zero-Sum Repeated Game Converse