Capacity and Zero-Error Capacity of the Chemical Channel with Feedback

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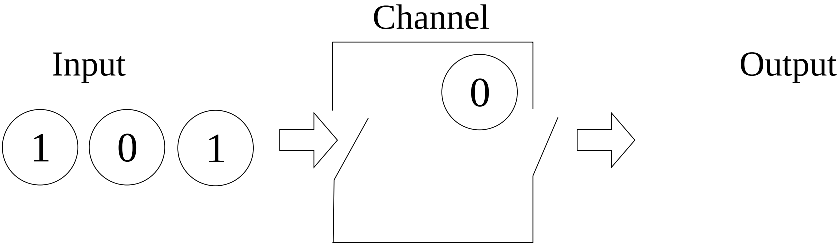
June 28, 2007

Focus of Talk

1. Chemical Channel Feedback Capacity
   - Numerical Calculations
   - Analytic Solution for Trapdoor Channel

2. Zero-error Communication Scheme

The Trapdoor Channel



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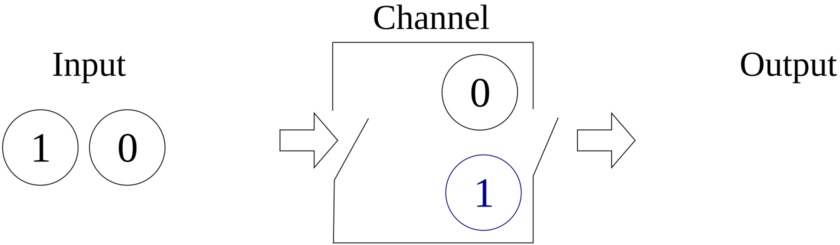
$s_0 = 0$

The Trapdoor Channel



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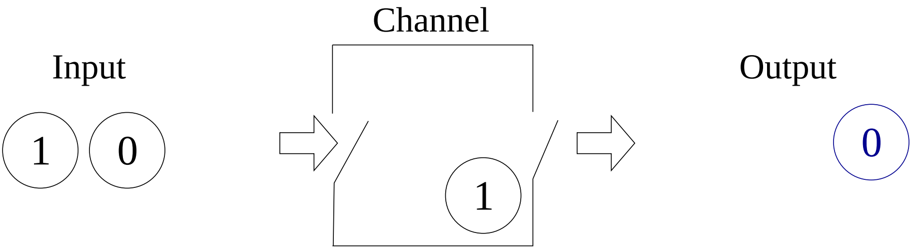

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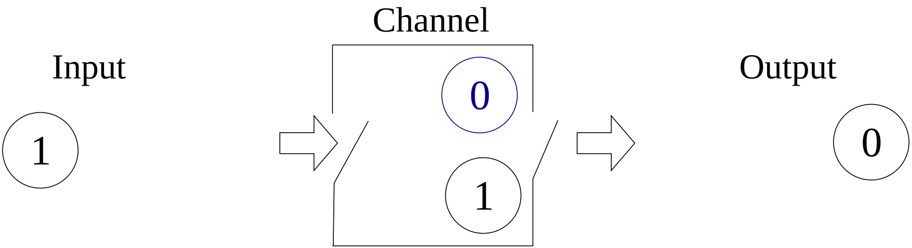

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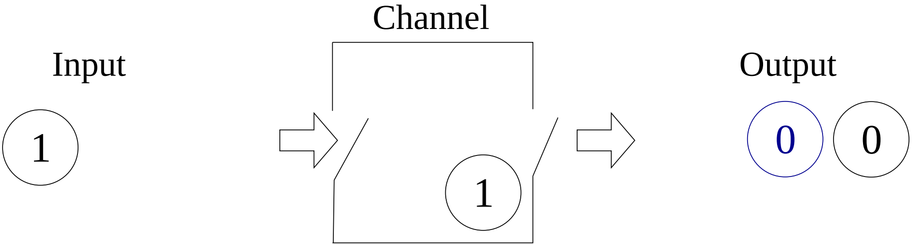

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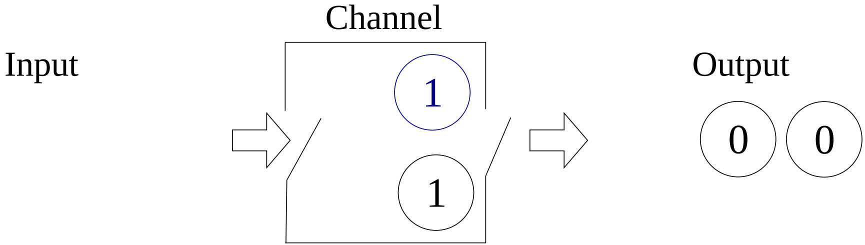

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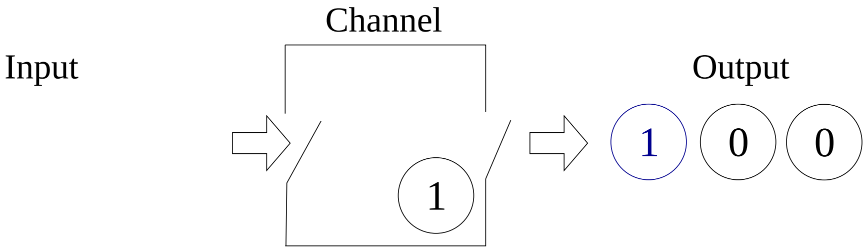

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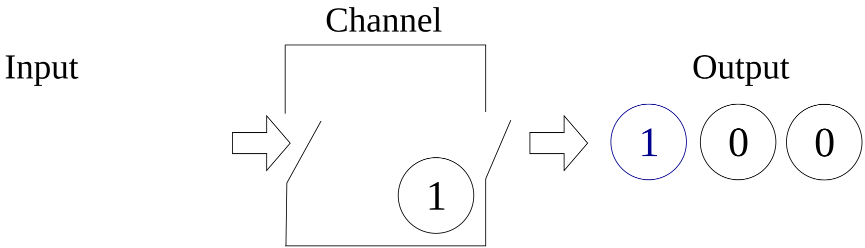

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Biochemical Interpretation [Berger 71]

The Chemical Channel

Balls are not equally likely to exit the channel.

x_t	s_{t-1}	$p(y_t = 0 \mid x_t, s_{t-1})$	$p(y_t = 1 \mid x_t, s_{t-1})$
0	0	1	0
0	1	p_1	$1 - p_1$
1	0	p_2	$1 - p_2$
1	1	0	1

Special cases:

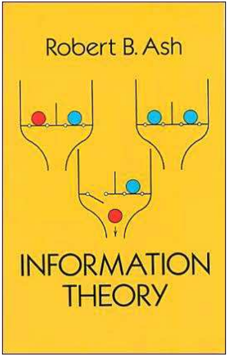
Trapdoor channel: $p_1 = p_2 = \frac{1}{2}$.

New vs. Old: $p_1 = 1 - p_2 = p_{switch}$.

'0' vs. '1': $p_1 = p_2 = p_{zero}$.

The Trapdoor Channel

Introduced by David Blackwell in 1961.
[Ash 65], [Ahlswede & Kaspi 87], [Ahlswede 98], [Kobayashi 02 & 03].



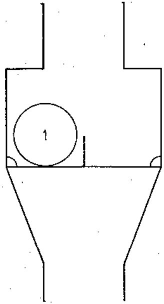


Fig. 7.1 A simple two-state channel.

A "simple two-state channel." - Blackwell

Ideas for Communication without Feedback

- Repeat each bit three time: $R = 1/3$ bit.

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- Repeat each bit twice: $R = 1/2$ bit. [Ahlswede & Kaspi 87]
- $C \approx 0.572$ bits per channel use. [Kobayashi & Morita 03]

Communication Setting (with feedback)

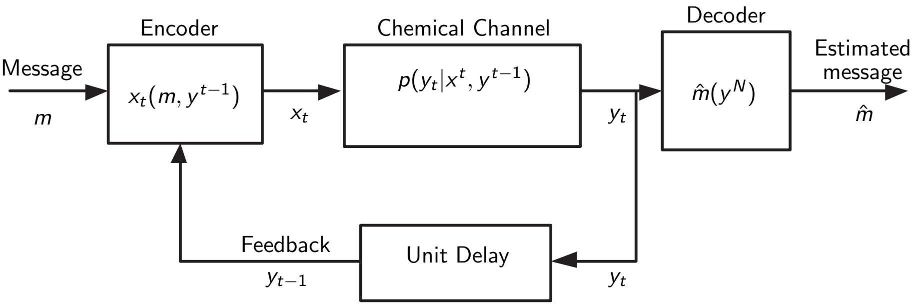

Figure: Communication with feedback

Feedback Capacity of FSC

Lower and upper bounds:

$$C_{FB} \geq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} \min_{s_0} I(X^N \to Y^N|s_0)$$

$$C_{FB} \leq \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} \max_{s_0} I(X^N \to Y^N|s_0)$$

[Permuter, Weissman & Goldmith ISIT06]

Directed Information

Mutual Information

$$I(X^n; Y^n) \quad = \quad \sum_{i=1}^{n} I(X^{n}; Y_i | Y^{i-1})$$

Directed Information was defined by Massey in 1990.

$$I(X^n \rightarrow Y^n) \quad \triangleq \quad \sum_{i=1}^{n} I(X^{i}; Y_i | Y^{i-1})$$

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Intuition [Massey 05]:

$$I(X^n; Y^n) = I(X^n \to Y^n) + I(Y^{n-1} \to X^n)$$

Feedback Capacity of Unifilar, Strongly Connected, FSC

Chemical channel has two other properties of interest.

1. Unifilar [Ziv 85]: State is deterministic function of past state, input, and output.
2. Strongly connected: Any state s_t can be reached with positive probability from any other state s_{t-1}.

Consequence

Initial state doesn't matter; upper and lower bounds become equal.

$$C_{FB} = \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_i|x^{i-1},y^{i-1})\}_{i=1}^N} I(X^N \to Y^N)$$

Feedback Capacity of Unifilar, Strongly Connected, FSC

$$
\begin{aligned}
C_{FB} &= \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}_{t=1}^{N}} I(X^N \to Y^N) \\
&= \lim_{N\to\infty} \frac{1}{N} \max_{\{p(x_t|x^{t-1},y^{t-1})\}_{t=1}^{N}} \sum_{t=1}^{N} I(X^t; Y_t|Y^{t-1}) \\
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&= \sup_{\{p(x_t|s_{t-1},y^{t-1})\}_{t\geq 1}} \liminf_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} I(X_t, S_{t-1}; Y_t|Y^{t-1})
\end{aligned}
$$

Dynamic Programming (infinite horizon, average reward)

Variable Assignments

State: $\beta_t = p(s_t|y^t)$

Action: $u_t = p(x_t|s_{t-1})$

Disturbance: $w_t = y_{t-1}$

Dynamic Programming Requirements

State evolution:

$$\beta_t \quad = \quad F(\beta_{t-1}, u_t, w_t)$$

Reward function per unit time:

$$g(\beta_{t-1}, u_t) \quad = \quad I(X_t, S_{t-1}; Y_t|\beta_{t-1})$$

Similar work: [Yang, Kavčić & Tatikonda 05], [Chen & Berger 05]

Dynamic Programming (infinite horizon, average reward)

Dynamic Programming Operator T

The dynamic programming operator T is given by

$$T \circ J(\beta) = \sup_{u \in \mathcal{U}} \left(g(\beta, u) + \int P_w(dw|\beta, u) J(F(\beta, u, w)) \right).$$

Bellman Equation

If there exist a function $J(\beta)$ and constant ρ that satisfy

$$J(\beta) = T \circ J(\beta) - \rho$$

then ρ is the optimal infinite horizon average reward.

Feedback Capacity of Chemical Channel (20 iterations)

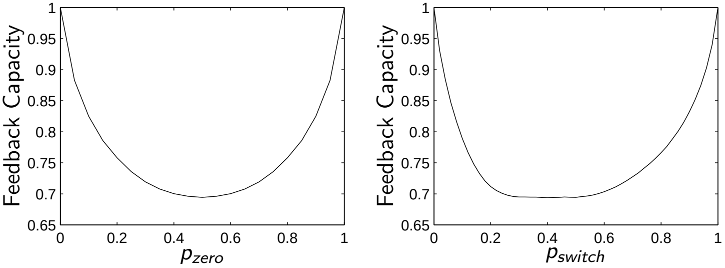

Figure: Feedback capacity of chemical channel as functions of two parameters.

Trapdoor channel feedback capacity found at $p_{zero} = 0.5$ and $p_{switch} = 0.5$.

Coincidence

Trapdoor channel: $C_{FB} \approx 0.694$ bits

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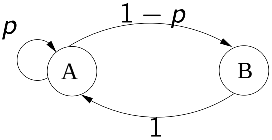
Homework Question

Entropy rate. Find the maximum entropy rate of the following two-state Markov chain:



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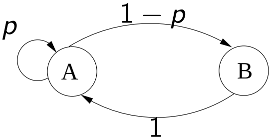

Solution: (Golden Ratio: $\phi = \frac{\sqrt{5}+1}{2}$)

$p^\star = \phi - 1 = \frac{1}{\phi}$

$H(\mathcal{X}) = \log \phi = 0.6942\ldots$ bits

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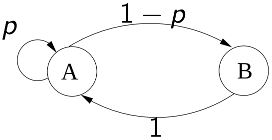

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Rename channel input: $\tilde{x}_t = x_t \oplus s_{t-1}$.
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$$x_{t-1} \oplus s_{t-2} = y_{t-1} \oplus s_{t-1}$$

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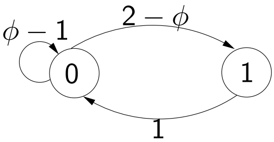
Use Markov input process

Case $\tilde{x}_t = 1$

$$\tilde{x}_t = 1 \quad \Rightarrow \quad \tilde{x}_{t-1} = 0$$



Decoding Example

Decoding rules

$$\tilde{x}_t = 0 \quad \Rightarrow \quad \tilde{x}_{t-1} = \tilde{y}_t$$
$$\tilde{x}_t = 1 \quad \Rightarrow \quad \tilde{x}_{t-1} = 0$$

$\tilde{x}^n$: 0

$\tilde{y}^n$: 1 1 0 1 0 0 1

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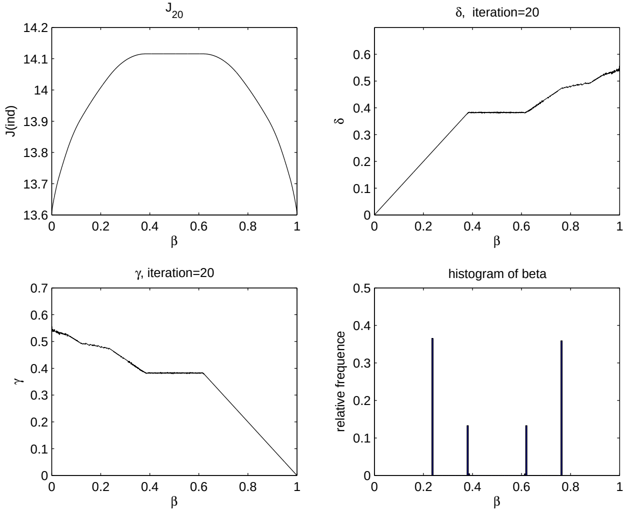
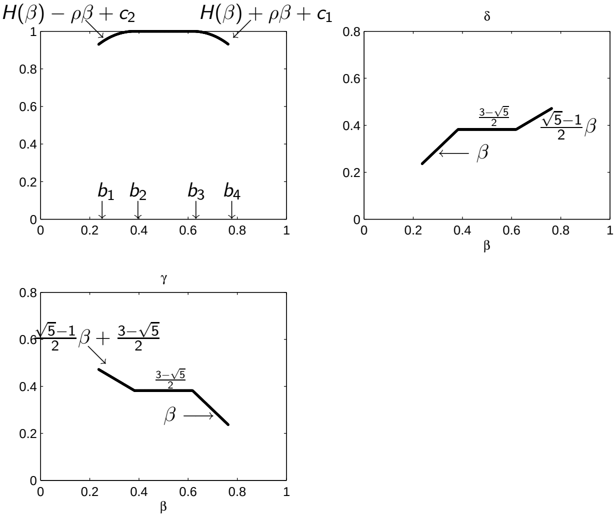
Dynamic Programming 20th Value iteration



Bellman Equation: $J(\beta) = T \circ J(\beta) - \rho.$

Conjectured Solution to the Bellman Equation



Bellman Equation: $J(\beta) = T \circ J(\beta) - \rho.$

Proven Feedback Capacity

Bellman Equation is satisfied.

Trapdoor channel feedback capacity:

$$C_{FB} = \log \phi = 0.6942\ldots \text{ bits}$$

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2. Zero-error Communication Scheme

A Zero-error Communication Scheme

$\tilde{x}^n$	flag	index
0010100010100101		

1. Message maps to unique sequence without repeating 1's.

A Zero-error Communication Scheme

$\tilde{x}^{n+1}$	flag	index
00101000101001010		

1. Message maps to unique sequence without repeating 1's.
2. Concatenate with 0.

A Zero-error Communication Scheme

$\tilde{x}^{n+1}$	flag	index
0010100010100101**0**	xxx	

1. Message maps to unique sequence without repeating 1's.
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3. Indicate if "inconsistency" was observed.

A Zero-error Communication Scheme

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4. If inconsistency exists, send index of inconsistency.

A Zero-error Communication Scheme

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n+1	3	$\log n / C_{FB}$

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Number of messages

How many binary sequences of length n without repeating 1's?
Fibonachi sequence: $f_n \doteq \phi^n$.

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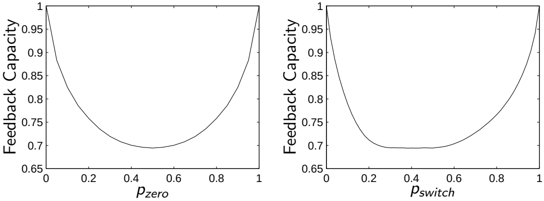
Conclusion

- Chemical Channel



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