# Consolidating Achievable Regions of Multiple Descriptions

Lei Zhao, Paul Cuff
Department of Electrical Engineering
Stanford University, Stanford, CA 94305
Email: {leiz,cuff}@stanford.edu

Haim Permuter
Department of Electrical Engineering
Ben-Gurion University of the Negev
Email: haimp@ee.bgu.ac.il

*Abstract*— **In this paper, some existing inner bounds of multiple description problem are investigated. We prove that the El Gamal-Cover region [1] is a subset of the Zhang-Berger (ZB) region [7]. Furthermore, the Venkataramani-Kramer-Goyal region [8] is equivalent to the Zhang-Berger region.**

## I. INTRODUCTION

Multiple description (MD) is a source-coding problem with one encoder and a variety of decoders. The encoder generates multiple descriptions of a source sequence. Each decoder receives a different subset of the descriptions. Figure 1 shows an MD problem with three decoders. Decoder 1 receives the upper branch of the message and estimates the source to achieve distortion $D_1$ and Decoder 2 receives the lower branch of the encoded message and estimates the source with distortion constraint $D_2$, while decoder 0 receives both descriptions to reach a distortion $D_0$. El Gamal and Cover provided an inner bound on the rate region for this problem [1]. The El Gamal-Cover (EGC) inner bound has been shown to be optimal for many important special cases of multiple descriptions. Ahlswede showed that in the case of no excess rate, i.e., $R_1+R_2 = R(D_0)$, the EGC bound is tight [4]. Equitz and Cover, and Rimoldi proved that in the case of successive refinement where $D_2 = \infty$, the EGC bound is tight [2], [3]. Ozarow proved that for a Gaussian source with squared error distoriton [5] the EGC bound is the rate region. Recently, Fu and Yeung proved that the EGC bound is tight when Decoder 1 or Decoder 2 performs a lossless reconstruction of a deterministic function of the source [6].

However, in [7], Zhang and Berger developed a new inner bound and used it to show that the EGC region is not optimal in general. For a particular binary example, the Zhang-Berger region contains points outside the EGC region. But it is not known if the Zhang-Berger (ZB) region strictly includes the EGC region. In [8], Venkataramani, Kramer, and Goyal combined techniques from both inner bounds and developed a new region. The Venkataramani-Kramer-Goyal (VKG) region includes both the ZB region and the EGC region. They noted in the paper that it is not known if the VKG region strictly includes the ZB region. The main contribution of this paper is showing that the ZB region includes all proposed regions for the multiple description problem.

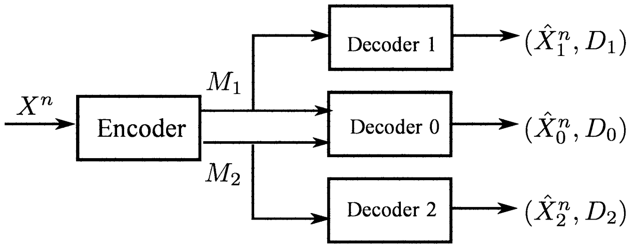

Fig. 1. The multiple description problem with three decoders.

### A. Problem setup and notations

Notation: Captital letter $X$ denotes a random variable and small letter $x$ denotes the corresponding realization or constant. Calligraphic letter $\mathcal{X}$ denotes the alphabet of $X$ and $|\mathcal{X}|$ denotes the cardinality of the alphabet. The subscripts in joint distributions are omitted. For example $p_{XY}(x,y)$ is written as $p(x,y)$. We use $X \perp Y$ to indicate that $X$ and $Y$ are independent, and $X - Y - Z$ to indicate that $X$ and $Z$ are conditionally independent given $Y$.

Let $X \in \mathcal{X}$ be an i.i.d source and $d_j(x,\hat{x}_j)$, $\hat{x}_j \in \hat{\mathcal{X}}_j$ for $j = 0,1,2$, be three distortion measures. The distortion between two sequences is defined as the average distortion

$$d_j(x^n, \hat{x}^n) = \frac{1}{n}\sum_{i=1}^{n} d_j(x_i, \hat{x}_i), \quad j = 0,1,2 \tag{1}$$

A $(2^{nR_1}, 2^{nR_2}, n)$ multiple description code consists of

1) One encoder: the encoder assigns two indices $m_1(x^n) \in [1 : 2^{nR_1})$ and $m_2(x^n) \in [1 : 2^{nR_2})$ to each source sequence $x^n \in \mathcal{X}^n$.
2) Three decoders: Decoder 1 assigns an estimate $\hat{x}_1^n$ to each index $m_1$. Decoder 2 assigns an estimate $\hat{x}_2^n$ to each index $m_2$. Decoder 3 assigns an estimate $\hat{x}_0^n$ to each pair $(m_1, m_2)$.

A rate pair $(R_1, R_2)$ is said to be achievable for the distortion triplet $(D_1, D_2, D_0)$ if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with average distortion

$$\limsup_{n\to\infty} E\left(d_j(X^n, \hat{X}_j^n)\right) \le D_j \text{ for } j = 0,1,2. \tag{2}$$

The rate distortion region $\mathcal{R}(D_1, D_2, D_0)$ is the closure of the set of achievable rate pairs $(R_1, R_2)$ for distortion triple

978-1-4244-4313-0/09/$25.00 ©2009 IEEE

$(D_1, D_2, D_0)$.

## II. THE EGC INNER BOUND

In the Shannon Theory Workshop 1979, El Gamal and Cover presented an achievable region for the multiple description problem. Let us name the region EGC* since it differs from the EGC region they published later.

**Theorem 1 ($\mathcal{R}_{\mathrm{EGC}^*}$ [9])** *A rate pair $(R_1, R_2)$ is achievable for multiple description for distortion triple $(D_0, D_1, D_2)$ if*

$$\begin{aligned} R_1 &\geq I(X;U_1), \\ R_2 &\geq I(X;U_2), \\ R_1 + R_2 &\geq I(X;U_1,U_2) + I(U_1;U_2); \end{aligned} \tag{3}$$

*for some $p(u_1,u_2|x)$ and deterministic functions $\phi_1, \phi_2, \phi_{12}$ such that*

$$\begin{aligned} E(d_1(X,\phi_1(U_1))) &\leq D_1 \\ E(d_2(X,\phi_2(U_2))) &\leq D_2 \\ E(d_0(X,\phi_{12}(U_1,U_2))) &\leq D_0. \end{aligned} \tag{4}$$

The EGC region published in 1982 [1] states

**Theorem 2 ( [1])** *A rate pair $(R_1, R_2)$ is achievable for multiple descriptions for distortion triple $(D_0, D_1, D_2)$ if*

$$\begin{aligned} R_1 &\geq I(X;\hat{X}_1), \\ R_2 &\geq I(X;\hat{X}_2), \\ R_1 + R_2 &\geq I(X;\hat{X}_0,\hat{X}_1,\hat{X}_2) + I(\hat{X}_1;\hat{X}_2); \end{aligned} \tag{5}$$

*for some $p(\hat{x}_0,\hat{x}_1,\hat{x}_2|x)$ such that*

$$E(d_j(X,\hat{X}_j)) \leq D_j, \text{ for } j = 0,1,2 \tag{6}$$

Remarks: It is readily seen that the EGC region includes the EGC* region. Both regions are not necessarily convex. They can be convexified by introducing a time-sharing random variables $Q$.

In the remainder of this section, we will prove that $\mathcal{R}_{\mathrm{EGC}} = \mathcal{R}_{\mathrm{EGC}^*}$.

**Lemma 1** *: For a given distribution $p(x,y)$, there exist random variables $Y$ and $W$, and a deterministic function $g$ such that $Y \sim p(y)$, $W \perp Y$ and $(g(W,Y),Y) \sim p(x,y)$. Furthermore, the cardinality of $W$ need not be larger than $(|\mathcal{X}|-1)|\mathcal{Y}|+1$.*

*Proof:* The idea is to use $W$ to simulate a channel from $Y$ to $X$. Consider $W \sim \text{Uniform}[0,1]$, $Y \sim p(y)$ and $Y \perp W$. Define $g(w,y) = F^{-1}_{X|Y=y}(w)$, where $F^{-1}_{X|Y=y}(w) = \inf_{x\in\mathbb{R}}\{F_{X|Y=y}(x) \geq w\}$. Then $(g(W,Y),Y)$ has the desired distribution. The proof uses the fact that any distribution can be generated from a uniform $[0,1]$ random variable.

If $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, the function $F_{X|Y=(\cdot)}(\cdot)$ has at most $(|\mathcal{X}|-1)|\mathcal{Y}|+1$ distinguished values. Arrange those values in increasing order $v_1, .., v_n$. Note that $0 < v_i < 1$, $\forall 1 \leq i \leq n-1$ and $v_n = 1$. Construct the probability mass function of a discrete random variable $W$ in the following way:

$$\begin{aligned} p(W = v_1) &= v_1, \\ p(W = v_i) &= v_i - v_{i-1}, 2 \leq i \leq n \end{aligned} \tag{7}$$

Then $(g(W,Y),Y)$ has the desired distribution. ■

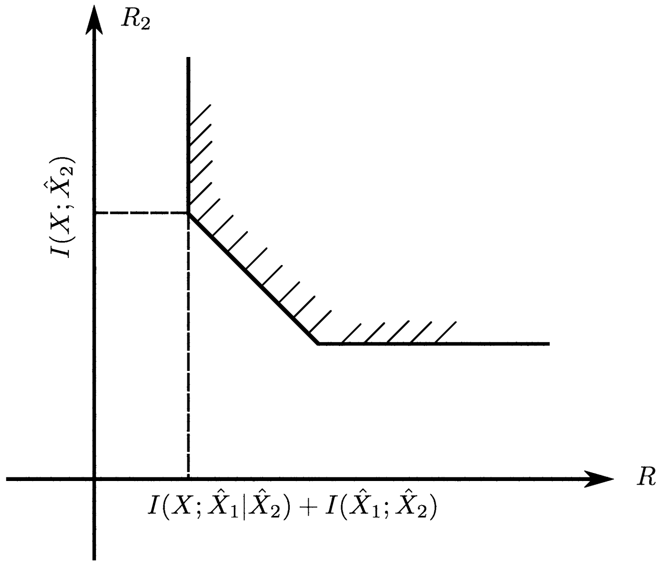

Fig. 2. A corner point in the EGC region with a fixed distribution.

**Theorem 3** $\mathcal{R}_{\mathrm{EGC}^*} = \mathcal{R}_{\mathrm{EGC}}$.

*Proof:* Fix a distribution in the EGC region $p(x)p(\hat{x}_1,\hat{x}_2,\hat{x}_0|x)$. Decompose it as $p(x|\hat{x}_1,\hat{x}_2,\hat{x}_0)p(\hat{x}_1,\hat{x}_2,\hat{x}_0)$.

Lemma 1 indicates that there exist random variables $(W,\hat{X}_1,\hat{X}_2)$ and a deterministic function $g$ such that $W \perp (\hat{X}_1,\hat{X}_2)$ and $(\hat{X}_1,\hat{X}_2,\hat{X}_0) \sim p(\hat{x}_1,\hat{x}_2,\hat{x}_0)$, where $\hat{X}_0 = g(W,\hat{X}_1,\hat{X}_2)$. The quadruplet $(W,\hat{X}_1,\hat{X}_2,\hat{X}_0)$ we just construct induces a conditional distribution $p(w|\hat{x}_1,\hat{x}_2,\hat{x}_0)$.

Define the quintuplet $(X,\hat{X}_1,\hat{X}_2,\hat{X}_0,W)$ according to the distribution
$p(x|\hat{x}_1,\hat{x}_2,\hat{x}_0)p(\hat{x}_1,\hat{x}_2,\hat{x}_0)p(w|\hat{x}_1,\hat{x}_2,\hat{x}_0)$.

In this way, we have shown that there exist random variables $(X,\hat{X}_1,\hat{X}_2,\hat{X}_0,W)$ such that

- $(\hat{X}_0,\hat{X}_1,\hat{X}_2,X) \sim p(\hat{x}_0,\hat{x}_1,\hat{x}_2,x)$,
- $W \perp (\hat{X}_1,\hat{X}_2)$,
- $X - (\hat{X}_0,\hat{X}_1,\hat{X}_2) - W$,
- $\hat{X}_0 = g(W,\hat{X}_1,\hat{X}_2)$ for some deterministic function $g$.
- $E(d_j(X,\hat{X}_j)) \leq D_j$, for $j = 0,1,2$,

where the last property comes from the fact that $p(\hat{x}_0,\hat{x}_1,\hat{x}_2,x)$ is a valid distribution in the EGC region. We check one corner point $(R_1,R_2)$ (shown in Fig. 2) in the EGC region under the fixed distribution:

$$\begin{aligned} R_2 &= I(X;\hat{X}_2), \\ R_1 + R_2 &= I(X;\hat{X}_1,\hat{X}_2,\hat{X}_0) + I(\hat{X}_1;\hat{X}_2), \end{aligned} \tag{8}$$

Let us first show that this corner point falls in $\mathcal{R}_{\mathrm{EGC}^*}$.

Define $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$. It is easy to show that $(X, U_1, U_2)$ satisfies the distortion constraints by using the fact that $E(d_j(X, \hat{X}_j)) \leq D_j$ for $j = 0, 1, 2$ and $\hat{X}_0 = g(W, \hat{X}_1, \hat{X}_2)$. Thus $(X, U_1, U_2)$ defines a valid joint distribution for the EGC* region.

Consider the corner point $(R_1', R_2')$ in EGC*:

$$\begin{aligned} R_2' &= I(X; U_2) \\ &= I(X; \hat{X}_2) \\ &= R_2 \end{aligned} \tag{9}$$

$$\begin{aligned} R_1' + R_2' =& I(X; U_1, U_2) + I(U_1; U_2) \\ =& I(X; W, \hat{X}_1, \hat{X}_2) + I(\hat{X}_1, W; \hat{X}_2) \\ \overset{(a)}{=}& I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1, W; \hat{X}_2) \\ \overset{(b)}{=}& I(X; W, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ =& I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(X; W | \hat{X}_1, \hat{X}_2, \hat{X}_0) \\ &+ I(\hat{X}_1; \hat{X}_2) \\ \overset{(c)}{=}& I(X; \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(\hat{X}_1; \hat{X}_2) \\ =& R_1 + R_2, \end{aligned} \tag{10}$$

where $(a)$ follows the fact $X_0 = g(W, \hat{X}_1, \hat{X}_2)$; (b) follows because $W \perp (\hat{X}_1, \hat{X}_2)$; (c) is because $X - (\hat{X}_1, \hat{X}_2, \hat{X}_0) - W$ .

The other corner point can be proved using similar arguments. Thus $\mathcal{R}_{\text{EGC}} \subseteq \mathcal{R}_{\text{EGC}^*}$. Therefore $\mathcal{R}_{\text{EGC}} = \mathcal{R}_{\text{EGC}^*}$ . ■

## III. THE ZB INNER BOUND

Zhang and Berger [7] improved on the EGC* achievable scheme with a different approach. They introduced a new variable $U_0$. Its purpose, however, is not to send a message, built on the first two messages $M_1$ and $M_2$, to Encoder 0. Instead, $U_0$ is used to coordinate the two messages, so that part of them will be identical. This redundancy can actually improve overall efficiency.

The achieveability scheme developed by Zhang and Berger [7] gives the following rate region

**Theorem 4 ($\mathcal{R}_{\text{ZB}}$ [7])** *: A rate pair $(R_1, R_2)$ is achievable for multiple descriptions for distortion triple $(D_0, D_1, D_2)$ if*

$$\begin{aligned} R_1 \geq& I(X; U_0, U_1), \\ R_2 \geq& I(X; U_0, U_2), \\ R_1 + R_2 \geq& I(X; U_0, U_1, U_2) + I(U_0; X) \\ &+ I(U_1; U_2 | U_0); \end{aligned} \tag{11}$$

*for some $p(u_0, u_1, u_2 | x)$ and deterministic functions $\phi_1, \phi_2, \phi_{12}$ such that*

$$\begin{aligned} E(d_1(X, \phi_1(U_0, U_1)) &\leq D_1 \\ E(d_2(X, \phi_2(U_0, U_2)) &\leq D_2 \\ E(d_0(X, \phi_{12}(U_0, U_1, U_2)) &\leq D_0 \end{aligned} \tag{12}$$

Remarks: The above region is convex.

**Theorem 5** $\mathcal{R}_{\text{EGC}} \subseteq \mathcal{R}_{\text{ZB}}$.

*Proof:* Set $U_0 = \emptyset$ in the ZB region, we get the EGC* region. Theorem 3 proved that the EGC* region is equal to the EGC region. Thus the ZB region includes EGC region. Furthermore, since the ZB region is convex, the convexification of the EGC region still falls within the ZB region. Thus $\mathcal{R}_{\text{EGC}} \subseteq \mathcal{R}_{\text{ZB}}$. ■

Venkataramani, Kramer, and Goyal recently developed the following rate region, which can be viewed as a combination of the ZB region and the EGC region.

**Theorem 6 ($\mathcal{R}_{VKG}$, [8])** *A rate pair $(R_1, R_2)$ is achievable for multiple description for distortion triple $(D_0, D_1, D_2)$ if*

$$\begin{aligned} R_1 &\geq I(X; \hat{X}_1, U), \\ R_2 &\geq I(X; \hat{X}_2, U), \\ R_1 + R_2 &\geq I(X; U, \hat{X}_1, \hat{X}_2, \hat{X}_0) + I(U; X) \\ &\quad + I(\hat{X}_1; \hat{X}_2 | U); \end{aligned} \tag{13}$$

*for some* $p(u, \hat{x}_1, \hat{x}_2, \hat{x}_0 | x)$ *such that*

$$\begin{aligned} E(d_0(X, \hat{X}_0)) &\leq D_0, \\ E(d_1(X, \hat{X}_1)) &\leq D_1, \\ E(d_2(X, \hat{X}_2)) &\leq D_2. \end{aligned} \tag{14}$$

Remark: The above region is convex. And $\mathcal{R}_{\text{EGC}} \subseteq \mathcal{R}_{\text{VKG}}$, $\mathcal{R}_{\text{ZB}} \subseteq \mathcal{R}_{\text{VKG}}$ [8]. We include the proof here for completeness. First, by setting $U = \emptyset$, the region becomes the EGC region. Second, for any $(U_0, U_1, U_2)$ in the ZB region, set $U = U_0$, $\hat{X}_1 = \phi_1(U_0, U_1)$, $\hat{X}_2 = \phi_2(U_0, U_2)$ and $\hat{X}_0 = \phi_{12}(U_0, U_1, U_2)$.

$$\begin{aligned} I(X; \hat{X}_1, U) &= I(X; \phi_1(U_0, U_1), U_0) \\ &\leq I(X; U_0, U_1), \\ I(X; \hat{X}_2, U) &\leq I(X; U_0, U_2), \\ I(X; \hat{X}_1 \hat{X}_2, \hat{X}_0) &\leq I(X; U_0, U_1, U_2), \\ I(\hat{X}_1; \hat{X}_2 | U_0) &\leq I(U_1; U_2 | U_0), \end{aligned} \tag{15}$$

Thus $\mathcal{R}_{\text{ZB}} \subseteq \mathcal{R}_{\text{VKG}}$.

Due to the results of Section II, it may not be a surprise that extending the ZB scheme with a final message to Encoder 0 is not necessary. We proved this in the next theorem.

**Theorem 7** $\mathcal{R}_{\text{ZB}} = \mathcal{R}_{\text{VKG}}$.

*Proof:* Note that we only need to prove $\mathcal{R}_{\text{VKG}} \subseteq \mathcal{R}_{\text{ZB}}$. Following the same steps as the proof for Theorem 3, we can show that for any distribution $p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$ from the VKG region., there exist random variables $(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X, W)$ such that

- $(U, \hat{X}_0, \hat{X}_1, \hat{X}_2, X) \sim p(u, \hat{x}_0, \hat{x}_1, \hat{x}_2, x)$
- $W \perp (U, \hat{X}_1, \hat{X}_2)$
- $X - (U, \hat{X}_0, \hat{X}_1, \hat{X}_2) - W$
- $\hat{X}_0 = g(W, U, \hat{X}_1, \hat{X}_2)$ for some deterministic function $g$
- $E(d_j(X, \hat{X}_j)) \leq D_j$, for $j = 0, 1, 2$.

We check one corner point $(R_1, R_2)$ in the VKG region under the fixed distribution:

$$
\begin{aligned}
R_2 &= I(X;\hat{X}_2,U),\\
R_1+R_2 &= I(X;U,\hat{X}_0\hat{X}_1,\hat{X}_2)+I(U;X) \qquad (16)\\
&\quad + I(\hat{X}_1;\hat{X}_2|U).
\end{aligned}
$$

We prove in the following that this corner point falls in $\mathcal{R}_{\mathrm{ZB}}$.

Set $U_0 = U$, $U_1 = (\hat{X}_1, W)$, $U_2 = \hat{X}_2$ in the ZB region. $(X, U_0, U_1, U_2)$ satisfy the distortion constrains due to the fact that $E(d_j(X,\hat{X}_j)) \le D_j$ for $j = 0, 1, 2$ and $\hat{X}_0 = g(W, U, \hat{X}_1, \hat{X}_2)$. Thus $(X, U_0, U_1, U_2)$ defines a valid joint distribution for the ZB region. Consider the corner point $(R_1', R_2')$ in ZB under this distribution:

$$
\begin{aligned}
R_2' =& I(X;U_0,U_2)\\
=& I(X;U,\hat{X}_2) \qquad (17)\\
=& R_2
\end{aligned}
$$

$$
\begin{aligned}
R_1'+R_2' =& I(X;U_0,U_1,U_2)+I(U_0;X)+I(U_1;U_2|U_0)\\
=& I(X;U,W,\hat{X}_1,\hat{X}_2)+I(U;X)+I(\hat{X}_1,W;\hat{X}_2|U)\\
\overset{(a)}{=}& I(X;U,W,\hat{X}_1,\hat{X}_2,\hat{X}_0)+I(U;X)\\
&\quad + I(\hat{X}_1,W;\hat{X}_2|U)\\
\overset{(b)}{=}& I(X;U,W,\hat{X}_1,\hat{X}_2,\hat{X}_0)+I(U;X)\\
&\quad + I(\hat{X}_1;\hat{X}_2|U)\\
=& I(X;U,\hat{X}_1,\hat{X}_2,\hat{X}_0)+I(X;W|U,\hat{X}_1,\hat{X}_2,\hat{X}_0)\\
&\quad + I(U;X)+I(\hat{X}_1;\hat{X}_2|U)\\
\overset{(c)}{=}& I(X;U,\hat{X}_1,\hat{X}_2,\hat{X}_0)+I(U;X)+I(\hat{X}_1;\hat{X}_2|U)\\
=& R_1+R_2,
\end{aligned}
\qquad (18)
$$

where $(a)$ follows the fact $X_0 = g(W, \hat{X}_1, \hat{X}_2)$; (b) follows because $W \perp (U, \hat{X}_1, \hat{X}_2)$; (c) is because $X - (U, \hat{X}_1, \hat{X}_2, \hat{X}_0) - W$ .

The other corner point can be proved using similar arguments. Thus $\mathcal{R}_{\mathrm{VKG}} \subseteq \mathcal{R}_{\mathrm{ZB}}$. Therefore $\mathcal{R}_{\mathrm{VKG}} = \mathcal{R}_{\mathrm{ZB}}$ . ■

## IV. CONCLUSION

In this paper, we proved that the inner bound developed by El Gamal and Cover at the Shannon Theory Workshop is equal to the one they published in 1982. As a consequence the Zhang-Berger region includes the El Gamal-Cover region. We also proved that the Zhang-Berger region is equal to the recent region shown by Venkataramani, Kramer, and Goyal.

## V. ACKNOWLEDGEMENT

The authors would like to thank Professor Abbas El Gamal for very helpful suggestions. Discussion with Hao Zou is also acknowledged. This work is supported by the National Science Foundation through the grants CCF-0515303 and CCF-0635318.

## REFERENCES

[1] A. El Gamal and T.M. Cover,"Achievable rates for multiple descriptions," *IEEE Trans. Inf. Theory,* vol. IT-28, no. 6, pp. 851-857, 1982.

[2] W.H.R. Equitz and T.M. Cover, "Successive refinement of information," *IEEE Trans. Inf. Theory,* vol. IT-37, no. 2, pp. 269-275, 1991.

[3] B. Rimoldi, "Successive refinement of information: Characterization of the achievable rates," *IEEE Trans. Inf. Theory,* vol. IT-40, no. 1, pp. 253-259, January 1994.

[4] R. Ahlswede, "The rate-distortion region for multiple descriptions without excess rate," *IEEE Trans. Inf. Theory,* vol. IT-31, no. 6, pp. 721-726, 1985.

[5] L. Ozarow, "On a source-coding problem with two channels and three receivers," *Bell System Tech. J.*, vol. 59, no. 10, pp. 1909-1921, 1980.

[6] F.W. Fu and R.W. Yeung, "On the rate-distortion region for multiple descriptions," *IEEE Trans. Inf. Theory,* vol. IT-48, pp. 2012-2021, July 2002.

[7] Z. Zhang and T. Berger, "New results in binary multiple descriptions," *IEEE Trans. Inf. Theory,* vol. IT-33, no. 4, pp. 502-521, 1987.

[8] R. Venkataramani, G. Kramer, and V. K. Goyal, "Multiple description coding with many channels," *IEEE Trans. Inf. Theory, vol.* IT-49, no. 9, pp. 2106-2114, September 2003.

[9] A. El Gamal and T.M. Cover, "Information Theory of Multiple Descriptions," presented at Shannon Theory Workshop, Mt. Kisco, N.Y., Sept. 1979.