Wiretap Channels with Random States

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Ziv Golfeld, Haim Permuter

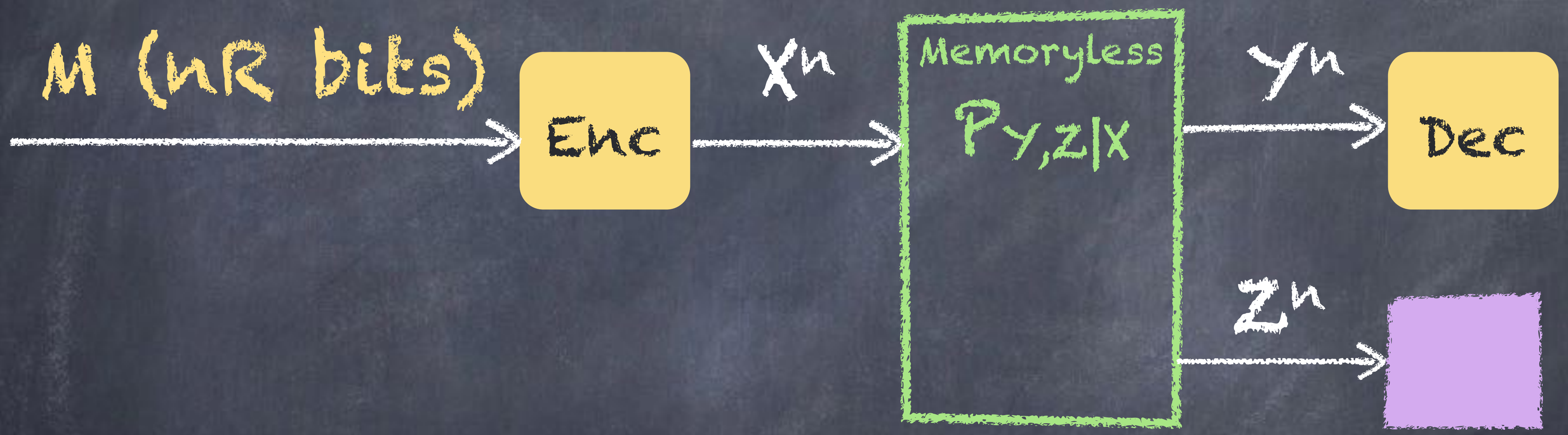
The Setting: 1975

- Wyner publishes five paper
- Two of interest in this talk
  - Wiretap Channel
  - Common Information

Wiretap Channel

- Foundation of physical-layer security

Wiretap Channel

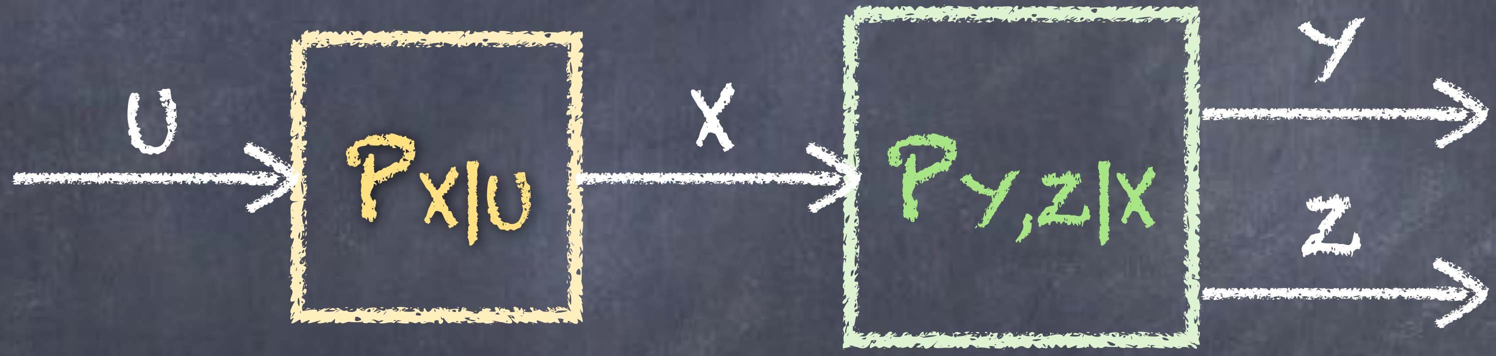


<u>Secrecy Capacity:</u>

- Reliable communication
- Z^n contains no information about M

Solution

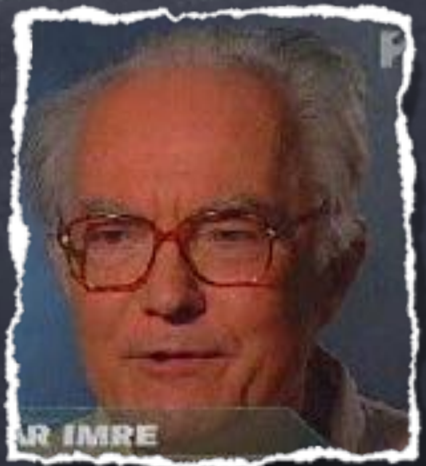
$$C_s = \max_{P_{X,U}} I(U;Y) - I(U;Z)$$



Solutions

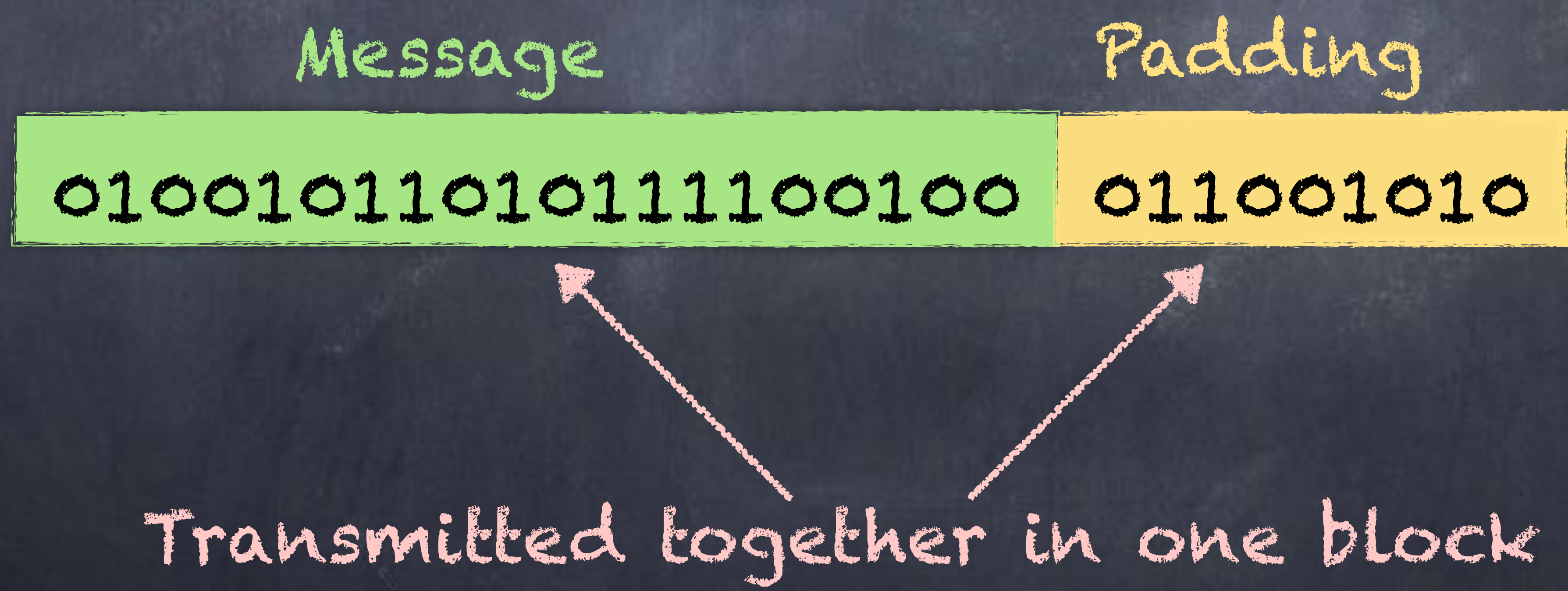
- 1975: Wyner introduced the problem and gave solution for degraded channels (U=X is sufficient)

- 1978: Csiszár and Körner gave solution for all channels

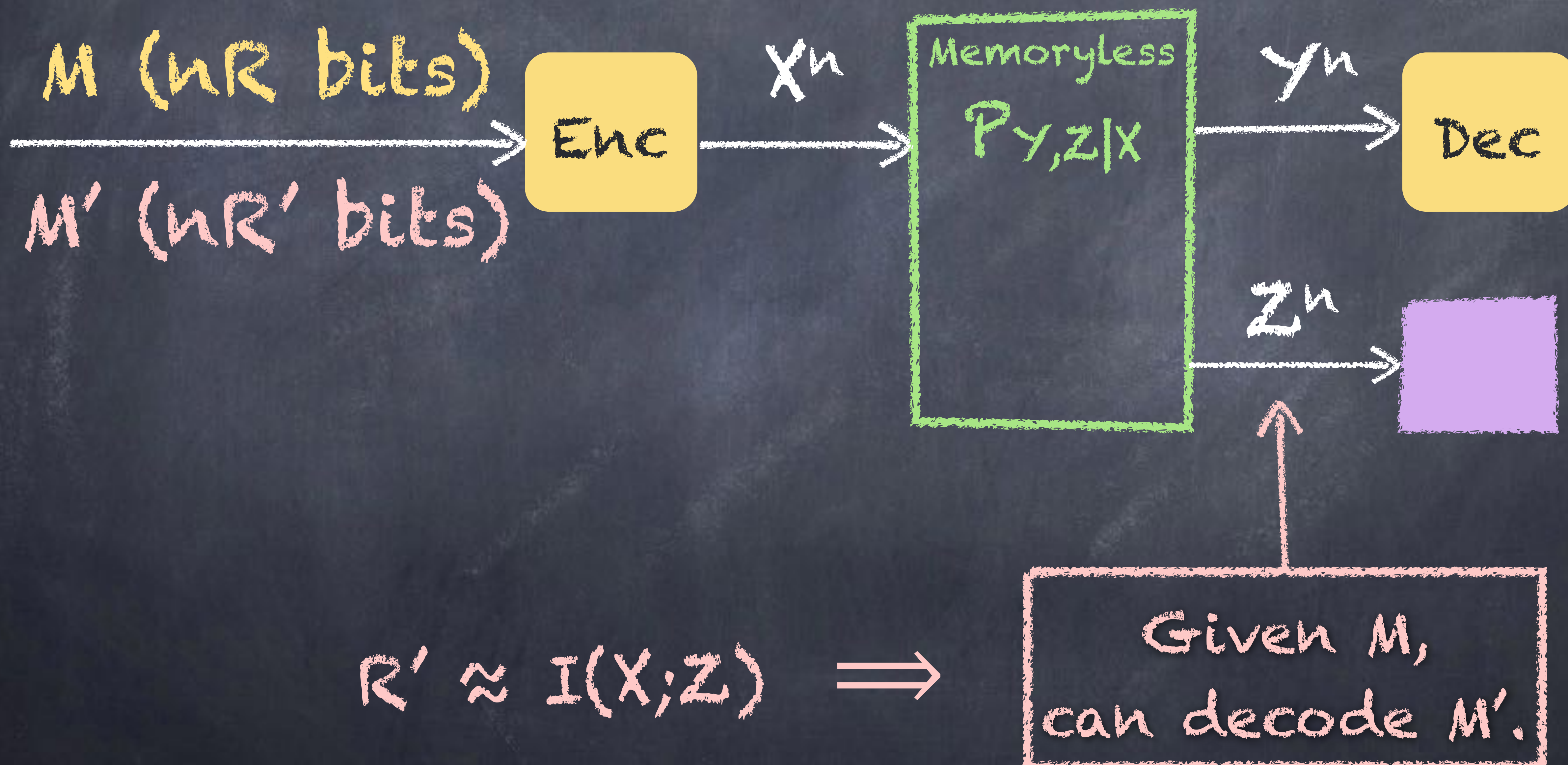


Encoding

- Random Codebook
- Pad with random garbage bits



Encoding Diagram



Wyner's security argument

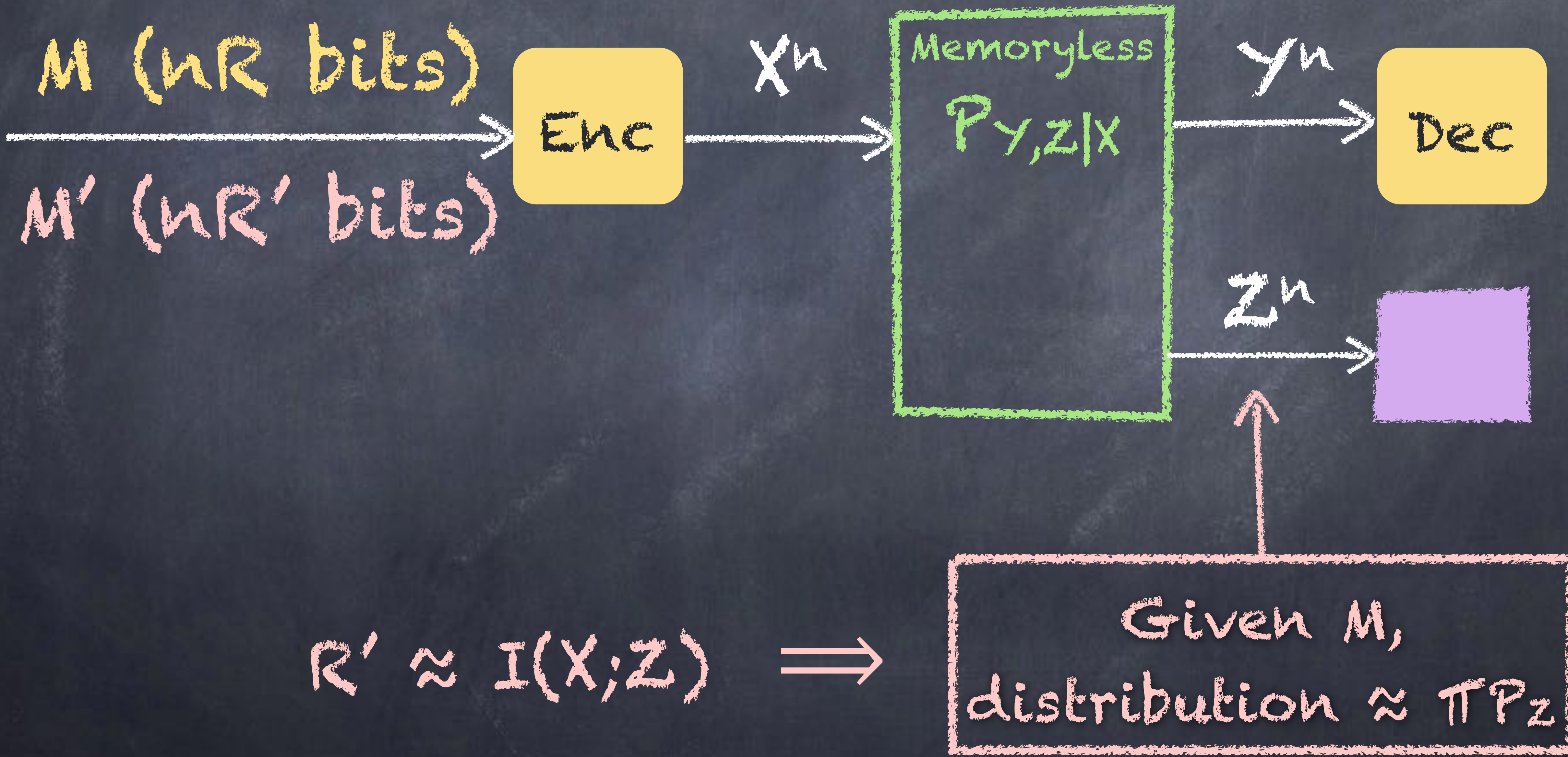
$$I(M,M';Z^n) = I(M;Z^n) + I(M';Z^n|M)$$

$$I(X^n;Z^n) \approx nI(X;Z)$$

$$H(M') = nR'$$

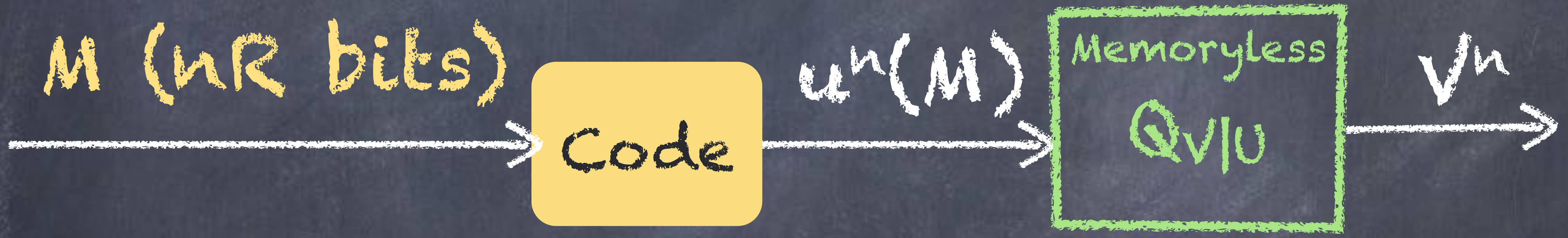
$$\boxed{\text{Decodable if } R'<I(X;Z)}$$

Encoding Concept



Soft Covering

- Theorem 6.3 of Wyner's C.I. paper:



Randomly select a codeword
Pass through a memoryless channel
Does induced output distribution match desired?

Output Distribution

Desired output distribution:

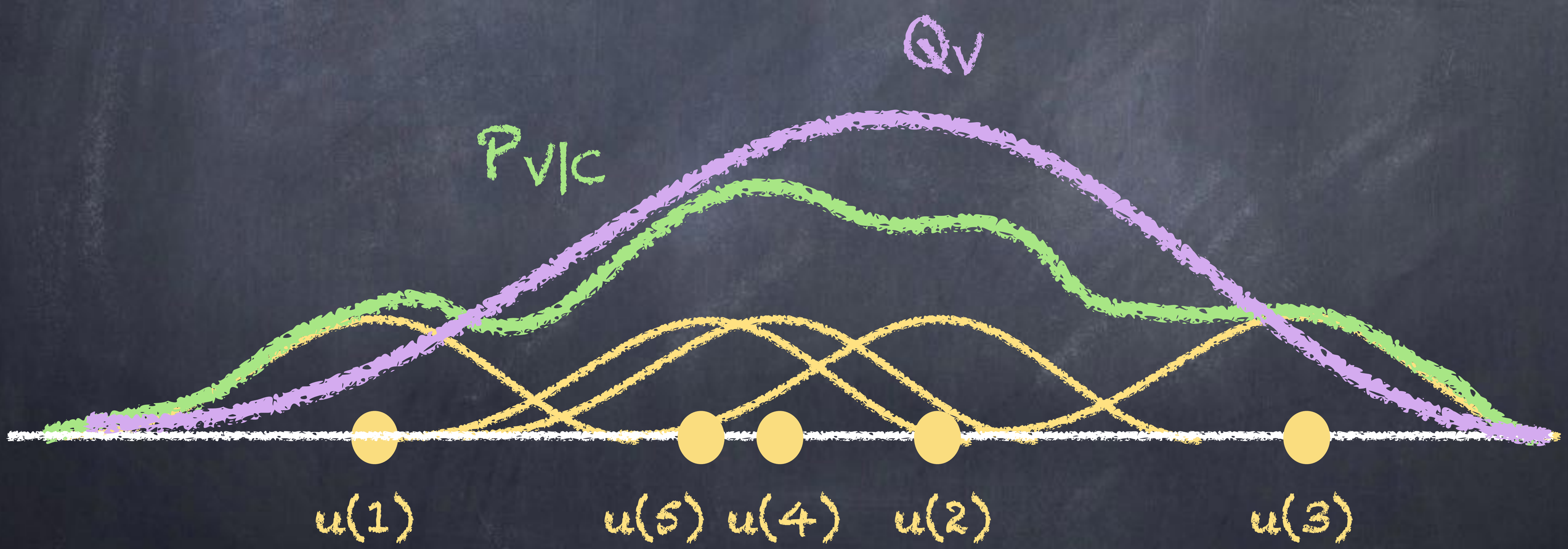
$$Q_V(v) = \sum_u Q_{V|U}(v|u) Q_U(u)$$

Induced output distribution:

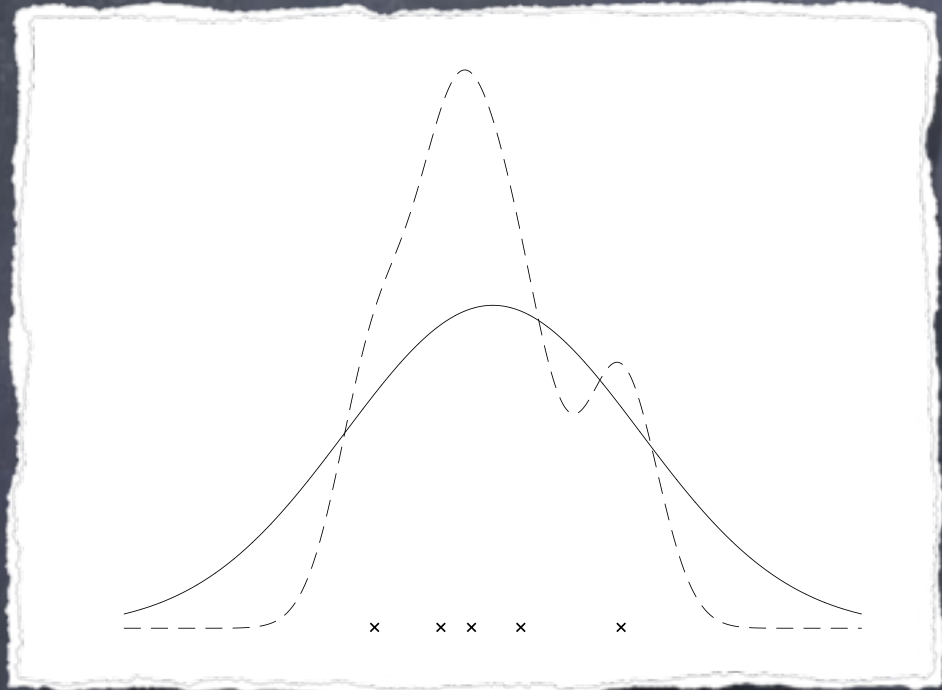
$$P_{V^n|\mathcal{C}} = 2^{-nR} \sum_{u^n(m)\in\mathcal{C}} Q_{V^n|U^n=u^n(m)}$$

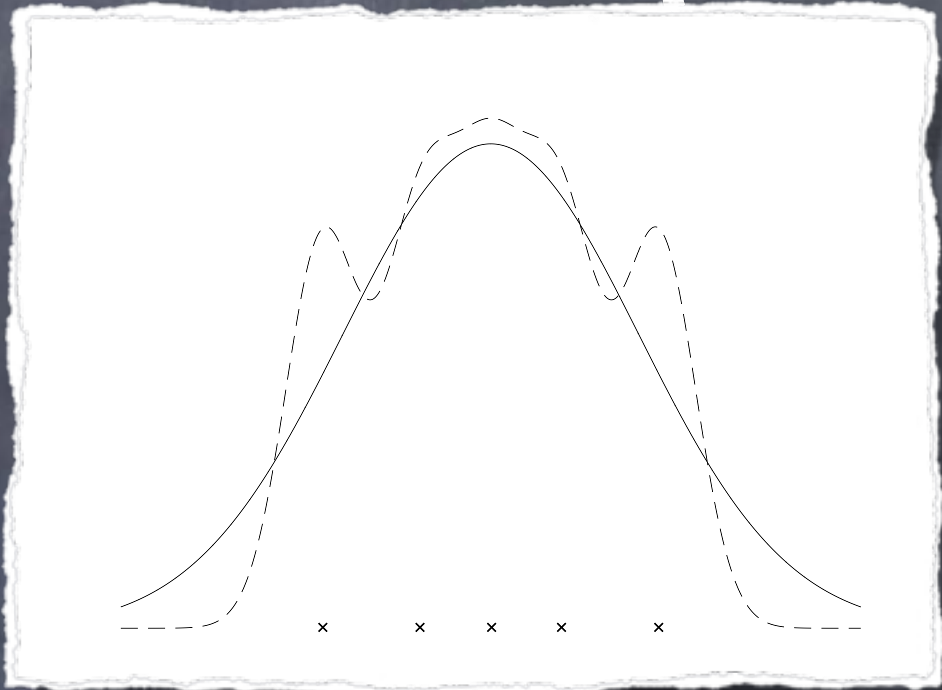
$$\boxed{\begin{aligned} Q_{V^n} &= \prod Q_V \\ Q_{U^n} &= \prod Q_U \\ Q_{V^n|U^n} &= \prod Q_{V|U} \end{aligned}}$$

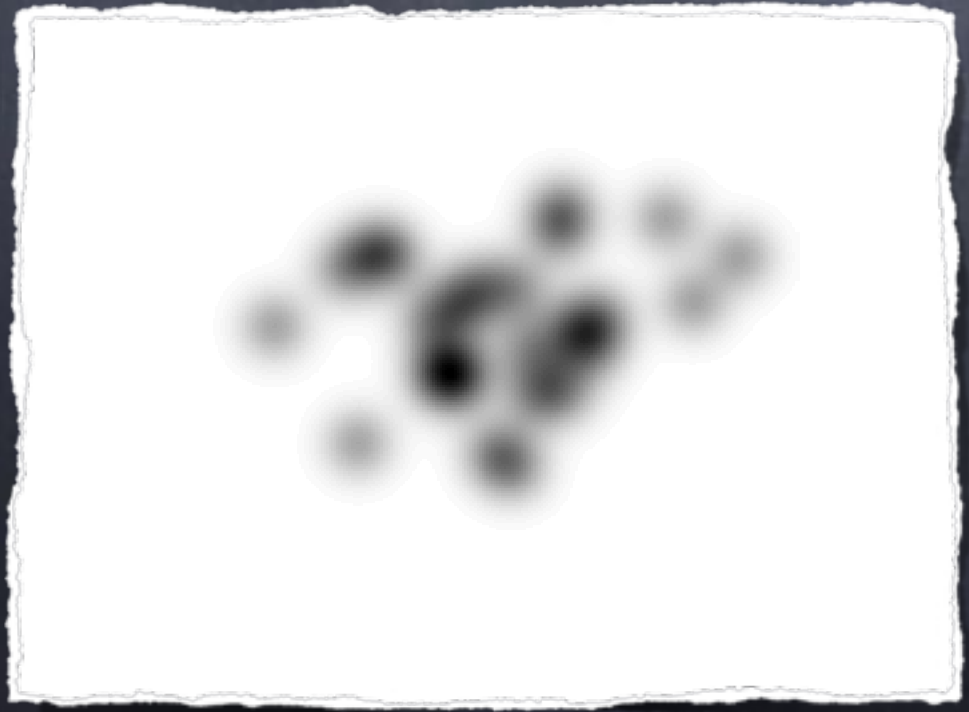
Output Distribution



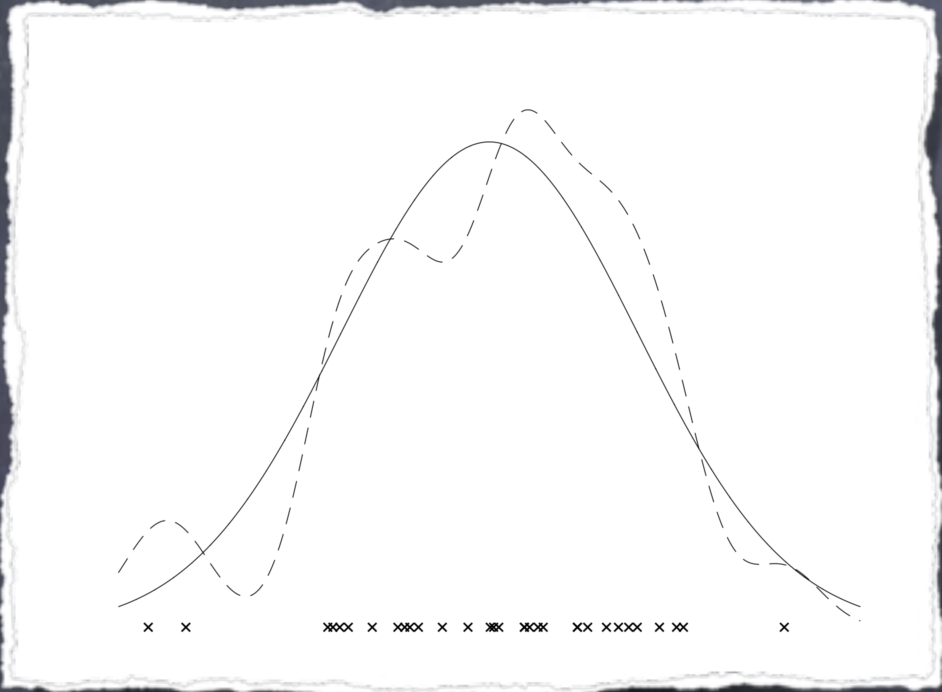
Gaussian Example

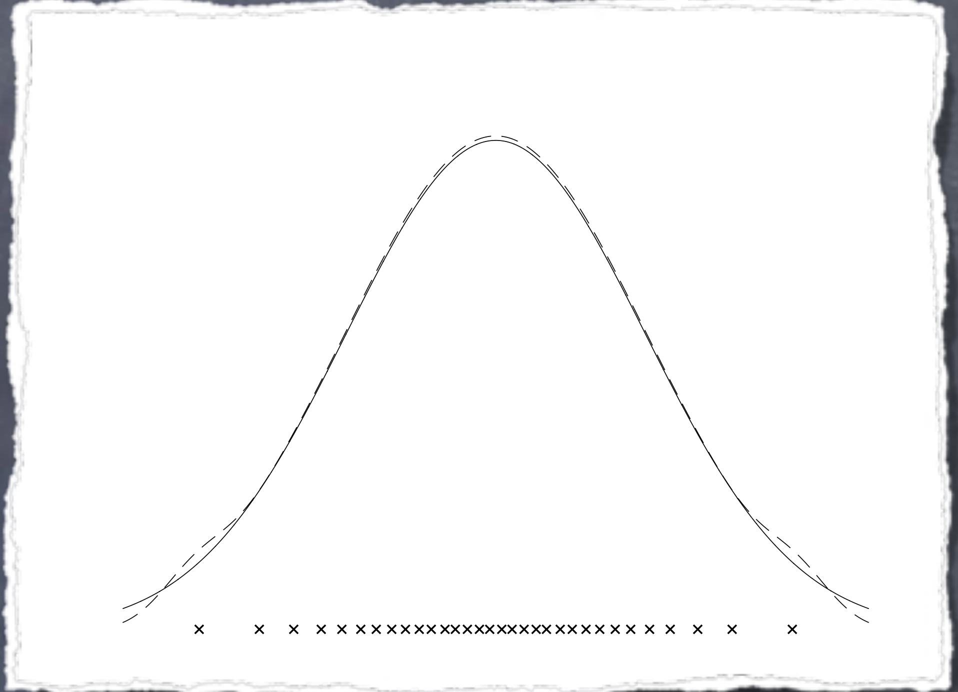


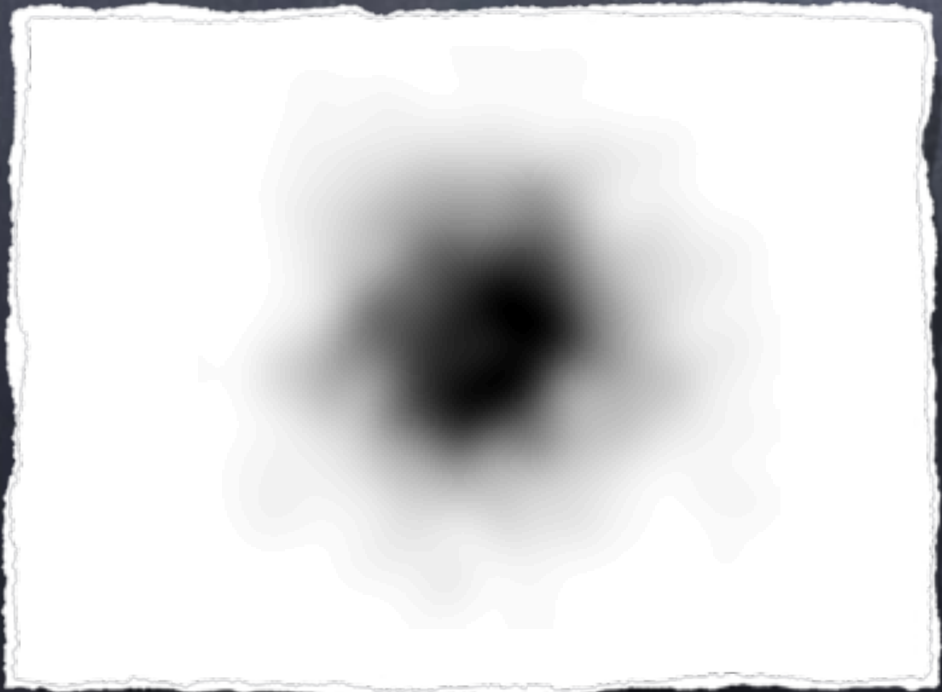




Gaussian Example





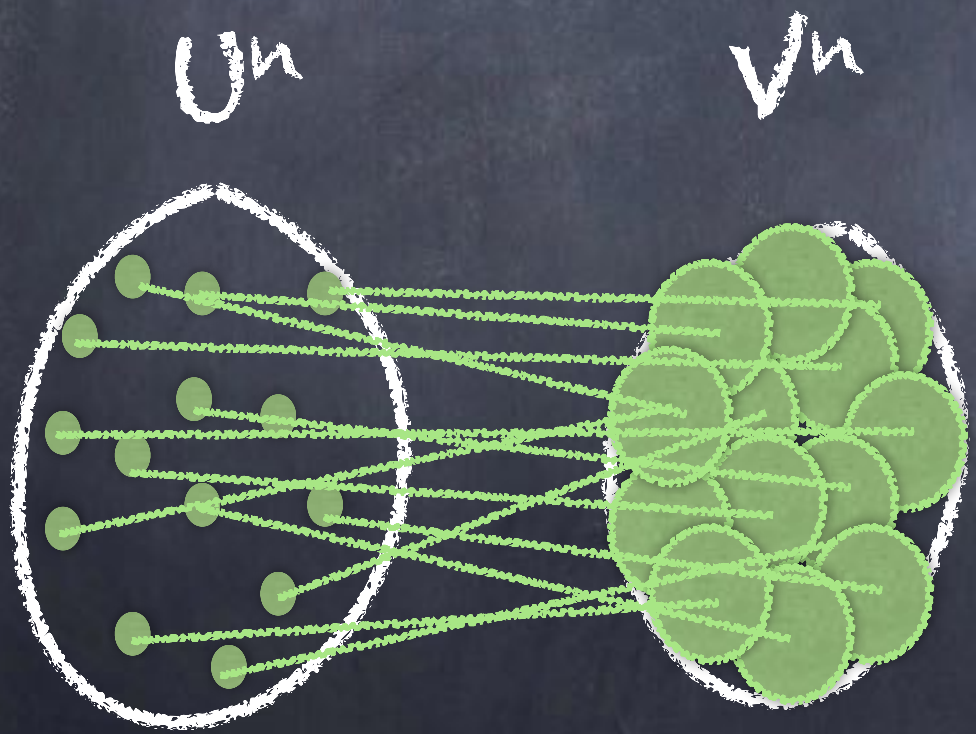


Soft Covering Lemma

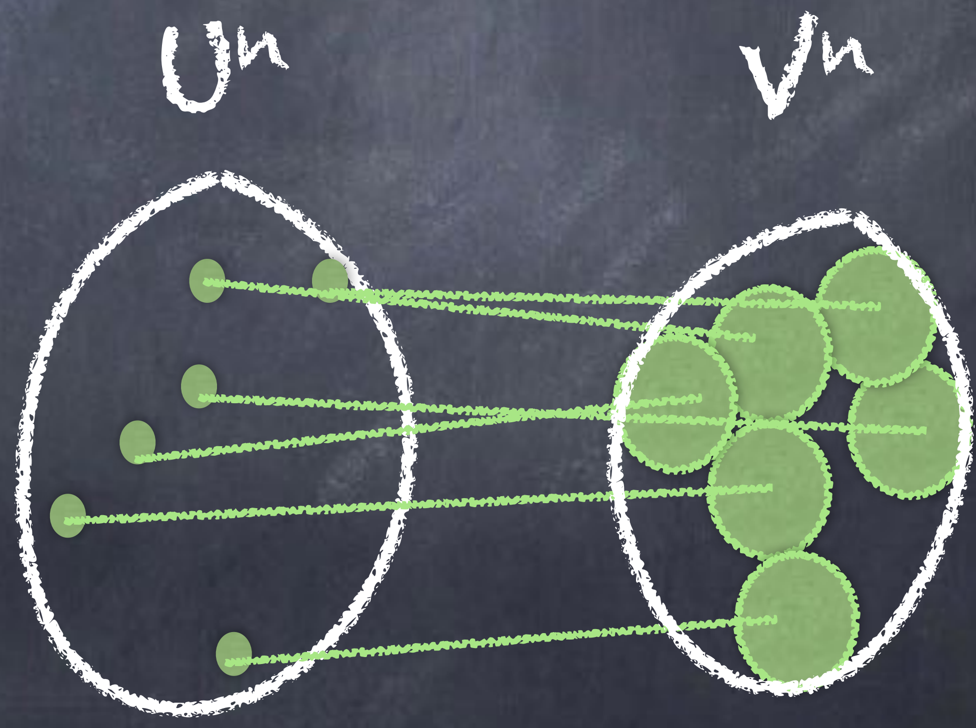
- Codebook size: If $R > I(U;V)$
- Codebook generation: $U^n(m) \sim Q_U$ i.i.d.
- Success: $P_{V^n|C} \approx Q_{V^n}$

Covering and Packing

Covering
(compression)



Packing
(transmission)



Covering

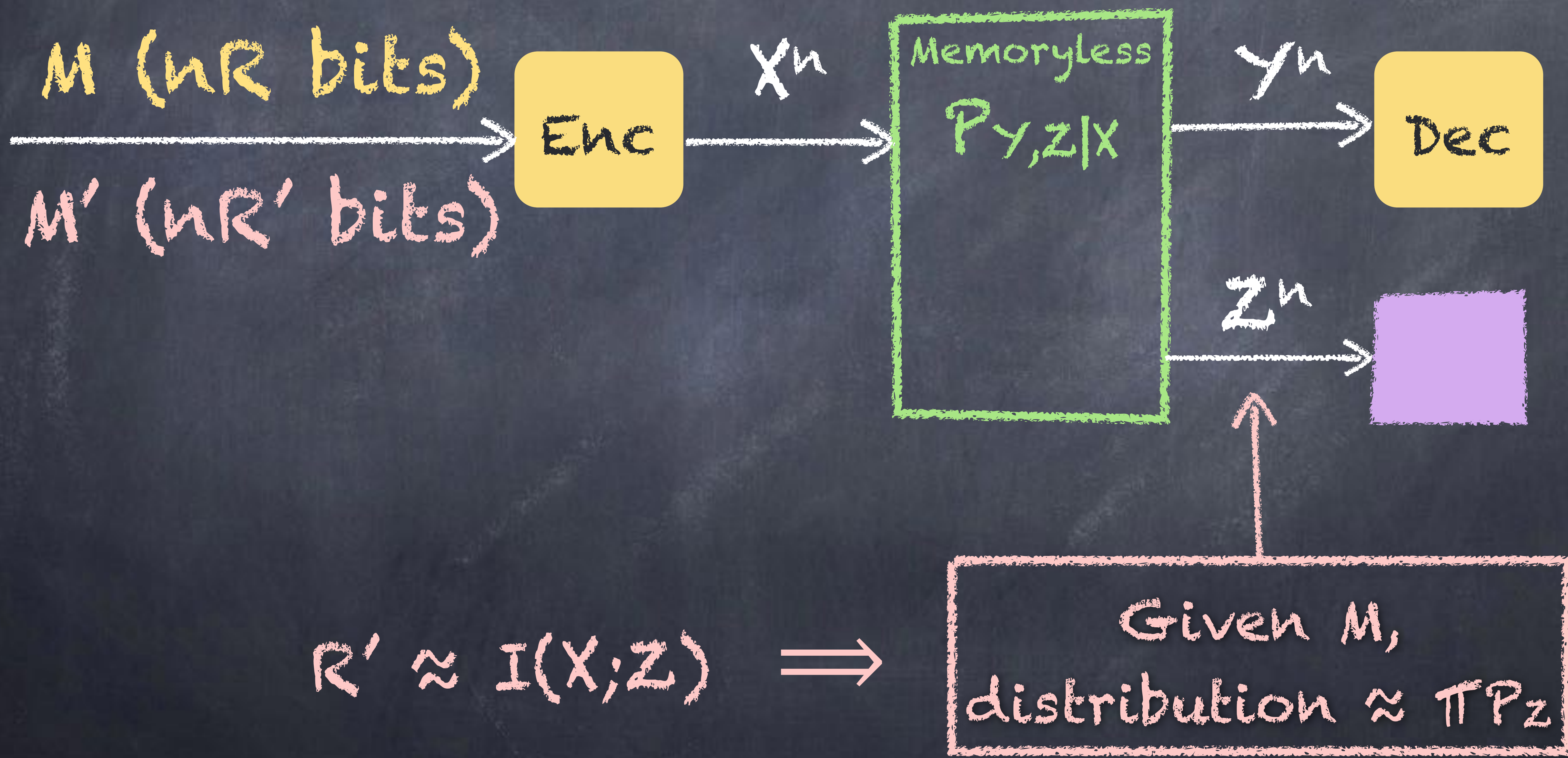
Hard covering:

$$\bigcup_{u^n(m)} \mathcal{T}_\epsilon(u^n(m)) \approx \mathcal{V}^n \quad \text{in probability}$$

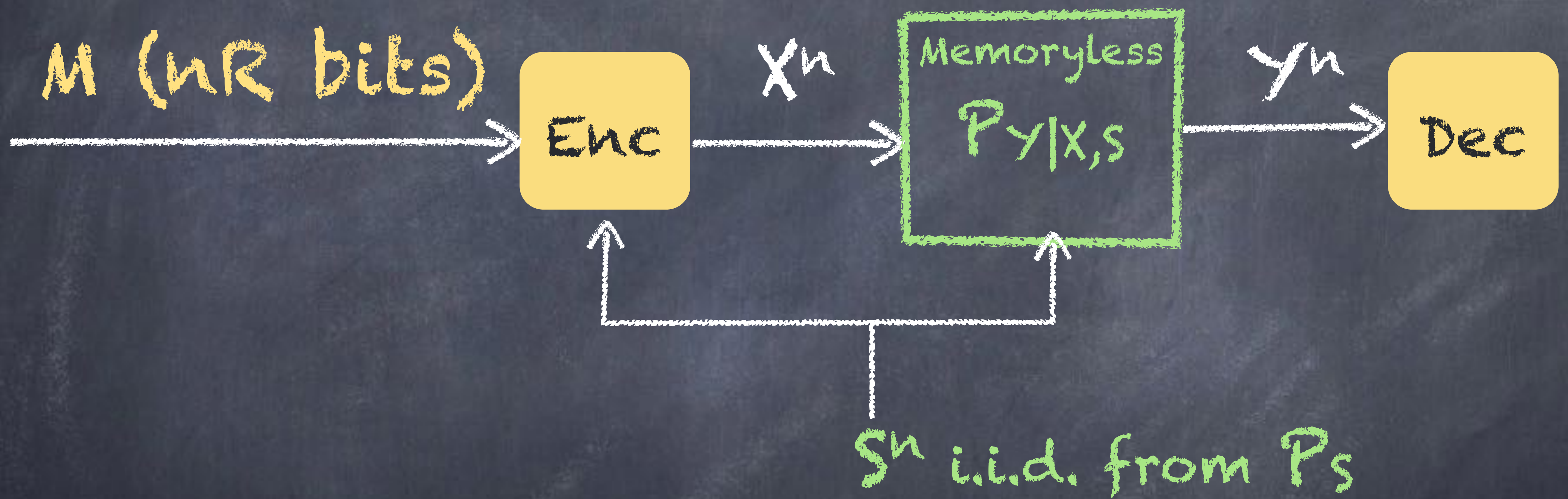
Soft covering:

$$2^{-nR} \sum_{u^n(m)} Q_{V^n|U^n=u^n(m)} \approx Q_{V^n}$$

Encoding Concept



Gelfand-Pinsker (state known to encoder)

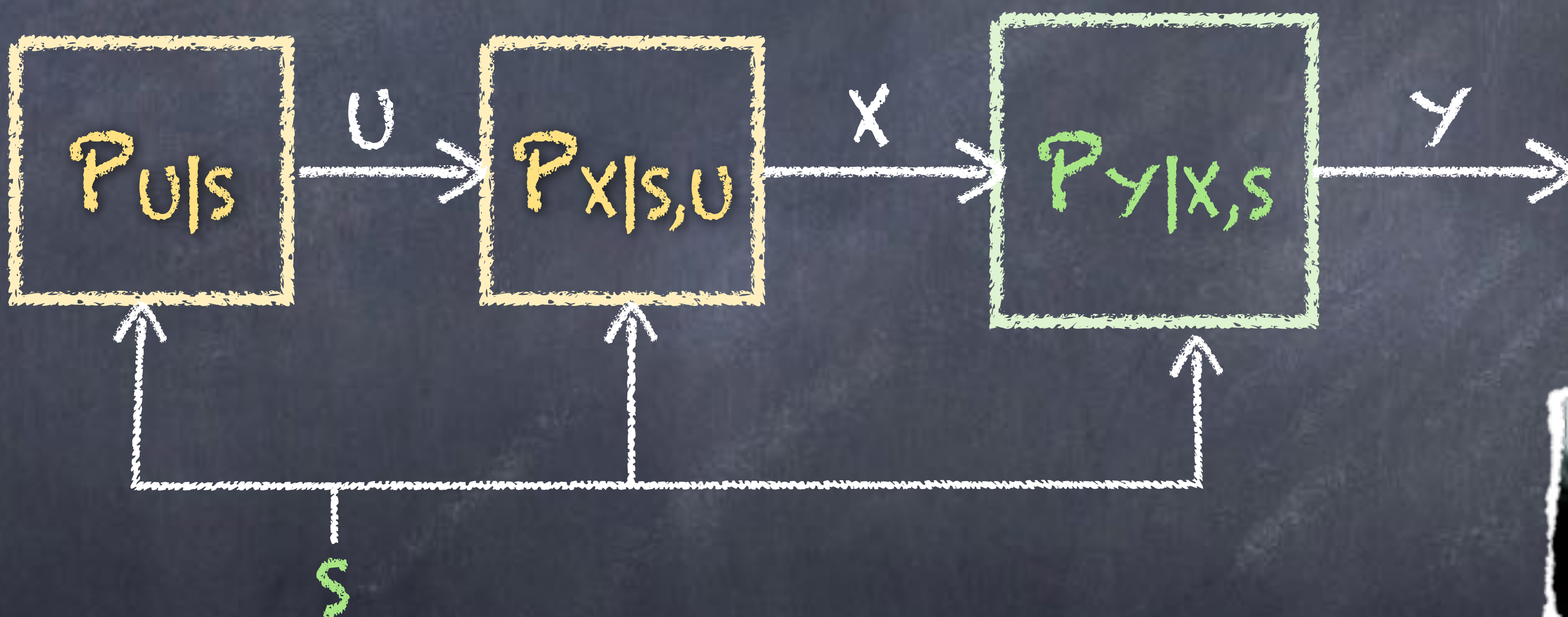


Capacity:

- Reliable communication

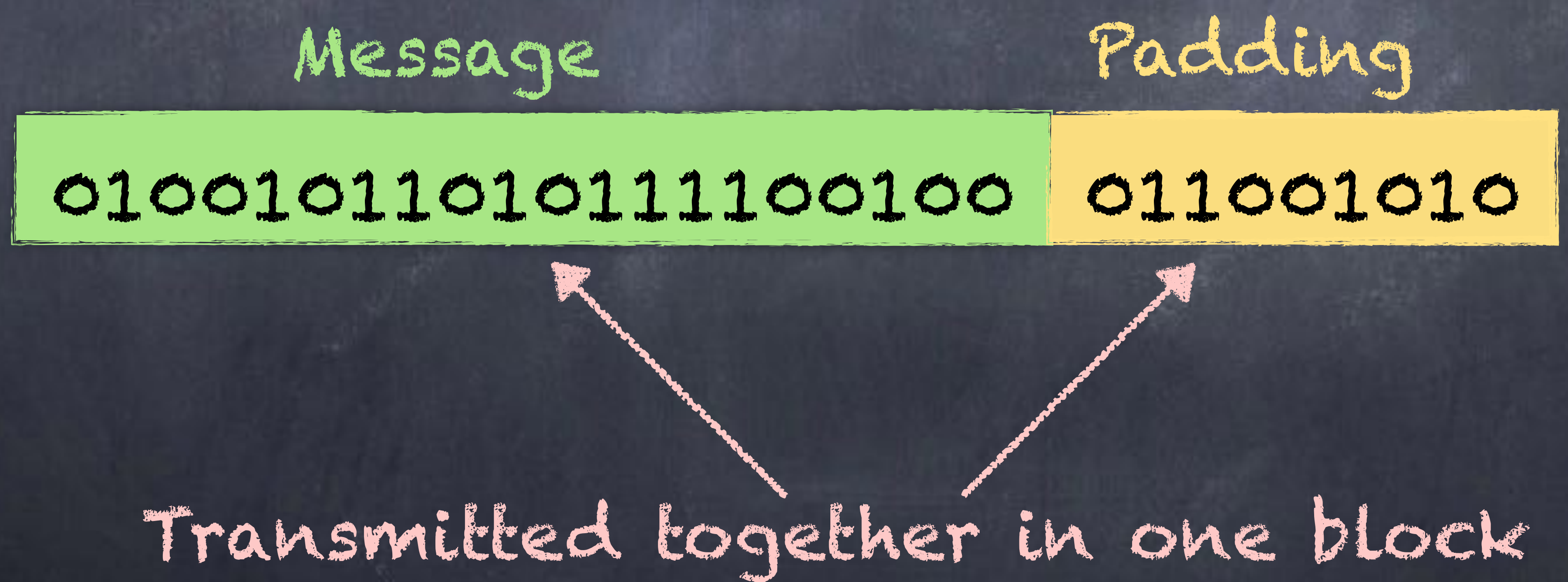
Solution (1980)

$$C = \max_{P_{X,U|S}} I(U;Y) - I(U;S)$$



Encoding

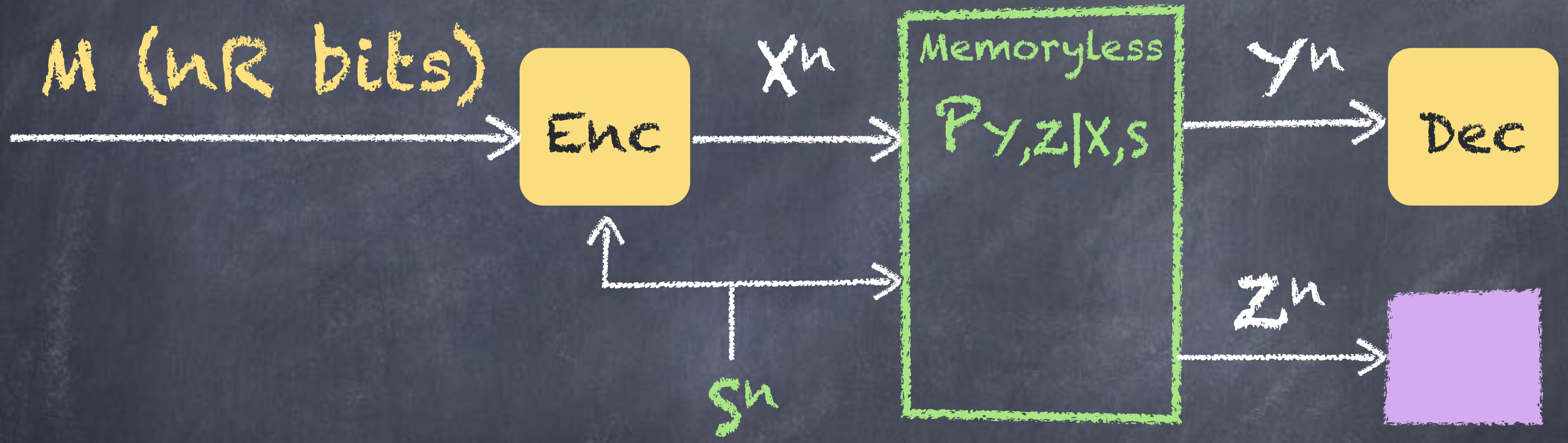
- Random Codebook
- Pad with skillfully chosen bits



Similarities

- Virtually the same
  - Same encoding
  - Same converse (except, iid S^n allows a skipped step)
  - Same problem statement:
    - Wiretap: M independent of Z^n
    - Gelfand-Pinsker: M independent of S^n

Wiretap Channel with State



Secrecy Capacity:

- Reliable communication
- Z^n contains no information about M

Same Encoding

$$C_s \geq \max_{P_{X,U|S}} I(U;Y) - \max \left\{ \begin{array}{c} I(U;Z), \\ I(U;S) \end{array} \right\}$$

Message	Padding
01001011010111100100	011001010

Transmitted together in one block

Chen and Han Vinck, 2006

Extract Key

Assume S is known to the intended receiver as well:

$$C_s \geq \max_{P_{X,U|S}} \min \left\{ \begin{array}{l} I(U;Y|S), \\ H(S|Z,U) \end{array} \right\}$$

Better in some cases!

Chia and El Gamal, 2012

Note: They consider causal state information.
This region is adapted to take advantage of non-causal state information.

Combined

Assume S is known to the intended receiver as well:

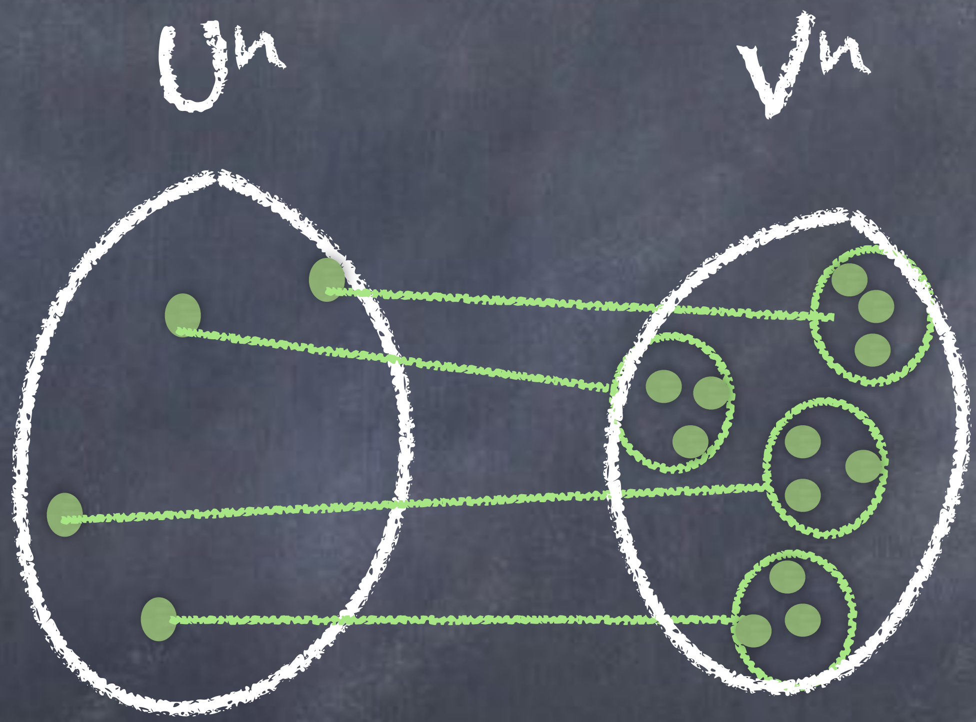
$$C_s \geq \max_{P_{X,U|S}} \min \left\{ \begin{array}{c} I(U;Y|S), \\ H(S|Z,U) + [I(U;Y,S) - I(U;Z)]_+ \end{array} \right\}$$

Chia and El Gamal, 2012

Note: They consider causal state information.
This region is adapted to take advantage of non-causal state information.

Our Scheme

Superposition code



U^n index is padding only
V^n index is message and padding

All secrecy comes from V
U^n is decoded by the eavesdropper

Our Scheme

$$C_s \geq \max_{P_{X,U,V|S} \,:\, I(U;Y) \geq I(U;S)} \min \left\{ \begin{array}{c} I(U,V;Y) - I(U,V;S), \\ I(V;Y|U) - I(V;Z|U) \end{array} \right\}$$

Can mimic Chia and El Gamal's key extraction by setting V=S

Beats previous regions

Other Related Work

- Prabhakaran, Eswaran, and Ramchandran, 2012:
  - Same superposition code but require U–V–(S,X) and U⊥S.
- Bassi, Bunin, Piantanida, and Shamai, 2016 (several papers):
  - Key generation and secure communication
  - Sources independent of channel
  - Generalized feedback

Simple Special case

- Unlimited public noise-free channel
- "Key Capacity with one-way communication"

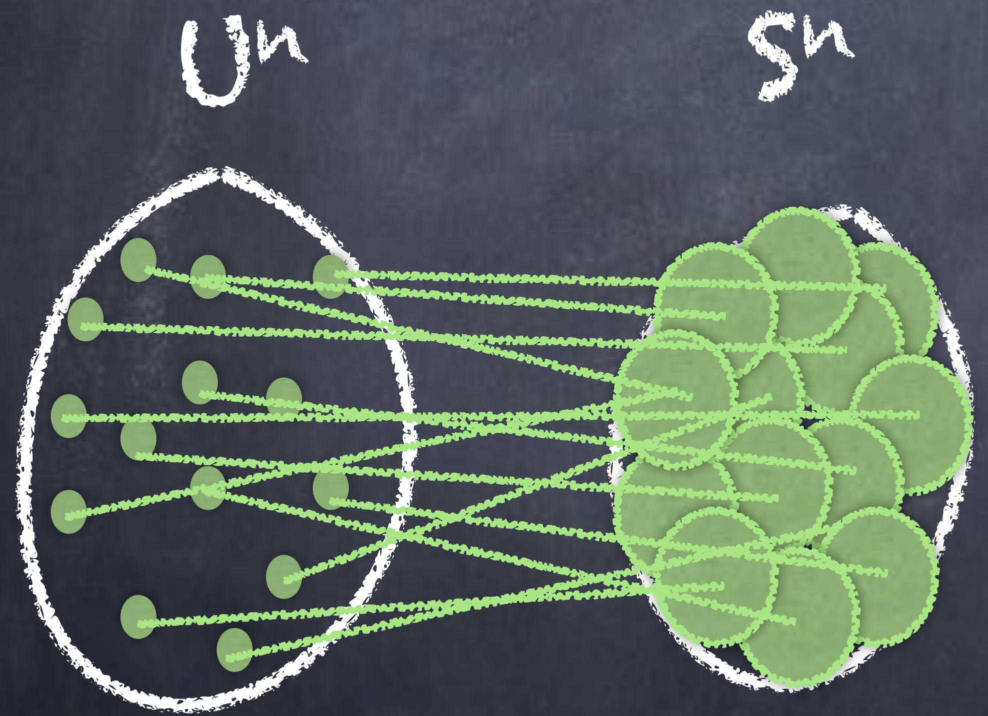
$$C_s = \max_{P_{U,V|S_x}} I(V;S_y|U) - I(V;S_z|U)$$

Achieved by our scheme

Apparently not by any others

Analysis Trick

- Likelihood encoder to choose padding



Soft Covering

Distribution 1:

- Choose U^n codeword uniformly at random
- Generate S^n memorylessly from U^n

Distribution 2:

- Use the above conditional distribution of U^n given V^n (this is the likelihood encoder)
- Let S^n be iid

Observation

- Lower layer of superposition code intended to be decoded by eavesdropper (i.e. "decoy")
  - Villard-Piantanida secure source coding
  - Key agreement with one way communication

Differential Privacy as a Mutual Information Constraint

Paul Cuff and Lanqing Yu

Database Privacy

- Let $X_1, X_2, \ldots, X_n$ be entries in a database
  - E.g. X_i is personal information about person i
- Let Y be the response to a query
  - We can denote Y_q to indicate that it depends on the query
- The job of the information provider is to answer queries and protect individual privacy

Differential Privacy

- ε-DP:
  - Let x and x′ differ in only one entry (i.e. $x_i = x'_i$ for all but one i)
  - $p(y|x) \leq e^{\varepsilon} p(y|x')$
- Why x and x′ differ in only one spot?
  - Convince someone to put their data in your database
- Why multiplicative constraint?
  - Posterior update is small

Dwork, 2006

A Common Technique

- Add Laplacean noise

Weaker DP

- (ε,δ)-DP:
  - $P(Y \in A|x) \leq e^{\varepsilon} P(Y \in A|x') + \delta$
- Additive Gaussian noise often provides privacy

Dwork, 2006

Mutual Information Differential Privacy

ε-MI-DP:

$$\max_{i, P_{X^n}} I(X_i; Y | X^{i-1}, X_{i+1}^n) < \epsilon$$

Claim

ε-DP > MI-DP > (ε,δ)-DP

Furthermore, if input or output alphabet is finite,

MI-DP = (ε,δ)-DP

similar to semantic security proof, Bellare, Tessaro, Vardy, 2012

Privacy Ordering

- α-DP > β-DP if for all β>0 there exists α such that α-DP $\Rightarrow$ β-DP.

Subadditivity of DP

- Multiple queries:
  - If k queries $Y_{q1}, Y_{q2}, \ldots, Y_{qk}$ each have differential privacy ε and are conditionally independent, the combined they have $k\varepsilon$ privacy.

Simple MI-DP Proof:

$$\begin{aligned} I(X;Y_1,Y_2) &= I(X;Y_1) + I(X;Y_2|Y_1) \\ &\leq I(X;Y_1) + I(X;Y_2) \end{aligned}$$

For clarity, conditioned database variables are omitted.

Common complaint

- Differentially privacy doesnt not mean that you cant learn about X_i.
  - Consider a database with correlated entries.

Simple MI-DP Explanation:

$$I(X_i;Y) \quad \not\leq \quad I(X_i;Y|X^{i-1},X_{i+1}^n)$$

Precise Bounds

(ϵ,δ)-Closeness

$$P \overset{(\epsilon,\delta)}{\approx} Q$$

if

$$\begin{aligned} P(A) &\leq e^{\epsilon} Q(A) + \delta, \quad \forall A \in \mathcal{F}, \\ Q(A) &\leq e^{\epsilon} P(A) + \delta, \quad \forall A \in \mathcal{F}. \end{aligned}$$

Special Cases

$$P \overset{(\epsilon,0)}{\approx} Q \quad \Longleftrightarrow \quad \left|\ln \frac{dP}{dQ}(a)\right| \leq \epsilon \quad \forall a \in \Omega.$$

$$P \overset{(0,\delta)}{\approx} Q \quad \Longleftrightarrow \quad \|P - Q\|_{TV} \leq \delta.$$

Relation to Detection Theory

$$P \overset{(\epsilon,\delta)}{\approx} Q$$

Kairouz, Oh, Viswanath, 2015

Tightest (ε,δ) Conversion

$$P \overset{(\epsilon,\delta)}{\approx} Q \implies P \overset{(\epsilon',\delta')}{\approx} Q.$$

for

$$\epsilon' \leq \epsilon$$

$$\delta' = 1 - \frac{\left(e^{\epsilon'} + 1\right)(1 - \delta)}{e^{\epsilon} + 1}$$

Simple Claim

$$P \overset{(\epsilon,0)}{\approx} Q \quad \Longrightarrow \quad \begin{aligned} D(P\|Q) &\le \epsilon \text{ nats}, \\ D(Q\|P) &\le \epsilon \text{ nats}. \end{aligned}$$

Tightest Claim

$$P \overset{(\epsilon,0)}{\approx} Q \quad \Longrightarrow \quad \begin{aligned} D(P\|Q) &\le \epsilon \frac{\left(e^{\epsilon}-1\right)\left(1-e^{-\epsilon}\right)}{\left(e^{\epsilon}-1\right)+\left(1-e^{-\epsilon}\right)} \text{ nats}, \\ D(Q\|P) &\le \epsilon \frac{\left(e^{\epsilon}-1\right)\left(1-e^{-\epsilon}\right)}{\left(e^{\epsilon}-1\right)+\left(1-e^{-\epsilon}\right)} \text{ nats}. \end{aligned}$$

Pinsker

$$D(P\|Q) \leq \epsilon \text{ nats} \quad \Longrightarrow \quad P \overset{\left(0,\sqrt{\epsilon/2}\right)}{\approx} Q.$$

Relative entropy to Mutual Information

If

$$D\left(P_{Y|X=x_1} \| P_{Y|X=x_2}\right) \leq \epsilon \quad \forall x_1, x_2 \in \mathcal{X}$$

then

$$I(X;Y) \leq \epsilon$$

Hint: Radius of information ball

Mutual Information to Total Variation

$$I(X;Y) \leq \epsilon \quad \Longrightarrow \quad \begin{array}{c} \|P_{Y|X=x_1} - P_{Y|X=x_2}\|_{TV} \leq \delta' \\ \forall x_1, x_2 \in \mathcal{X} \end{array}$$

$$\begin{aligned} \delta' &= 1 - 2h^{-1}(\ln 2 - \epsilon) \\ &\leq \sqrt{2\epsilon} \end{aligned}$$

Tightest bound, achieved with binary channel

Finite Alphabet

$$\left\| P_{Y|X=x_1} - P_{Y|X=x_2} \right\|_{TV} \leq \delta \quad \forall x_1, x_2 \in \mathcal{X} \implies I(X;Y) \leq \epsilon'$$

$$\epsilon' = 2h(\delta) + 2\delta \ln \left(\min \left\{ |\mathcal{Y}|, \max_i |\mathcal{X}_i| + 1 \right\} \right)$$

harder step

Continuity of entropy

Continuity of conditional entropy

inspired by Alicki and Fannes, 2004

Observation

$$\max_{P_{X^n}} I(X_i;Y|X^{i-1},X_{i+1}^n) = \max_{\prod_{t=1}^n P_{X_t}} I(X_i;Y) \quad \forall i$$

Either could be used for definition of MI-DP

Pointed out by Thomas Steinke