Communication in Networks for Coordinating Behavior

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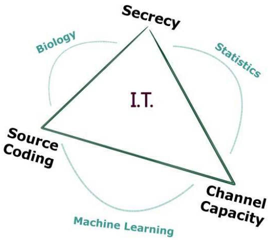
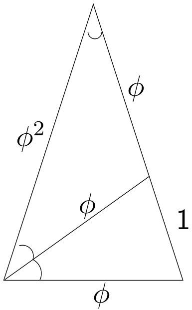
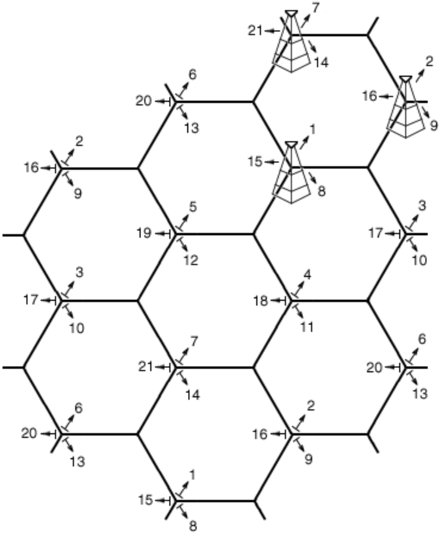
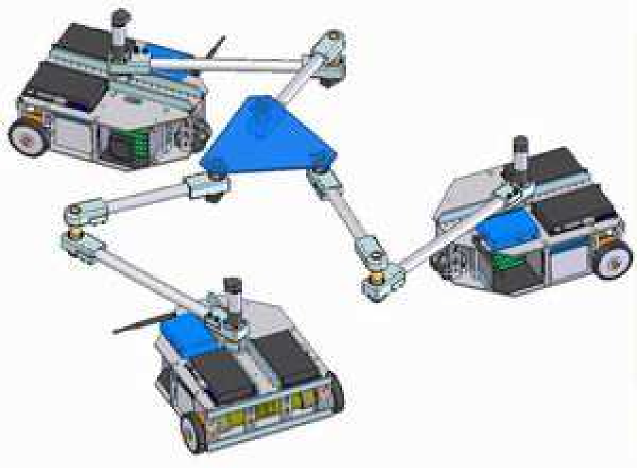




Cellular Communication Systems

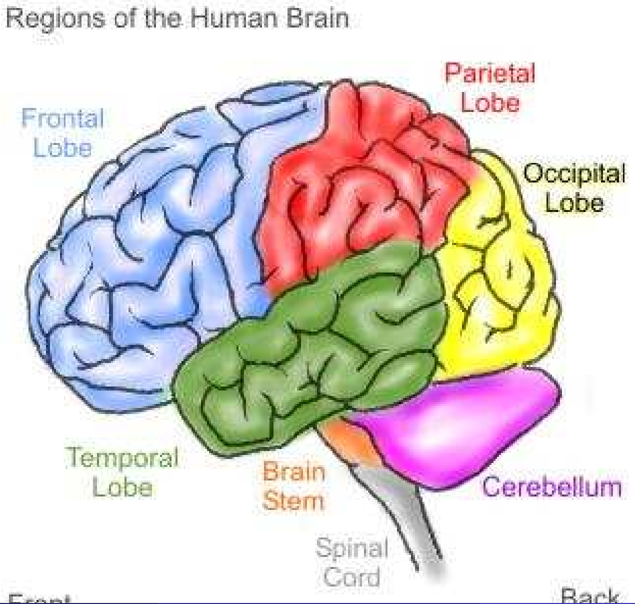


Distributed Sensing and Control



Competitive Settings

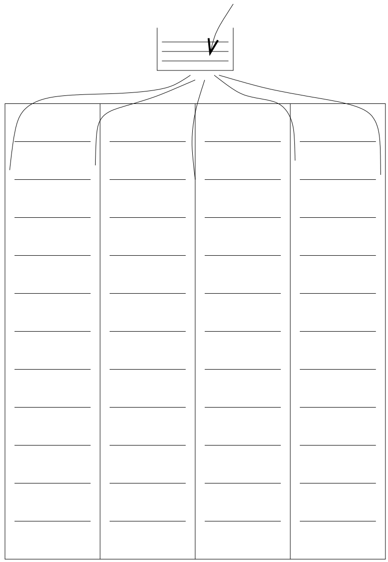
Parallel Processing in the Brain



Data Centers

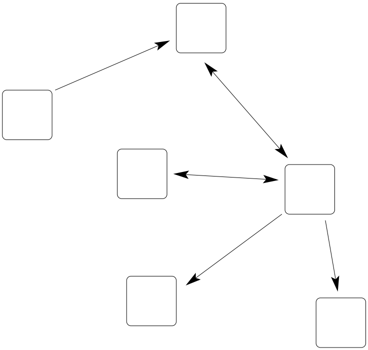
Data Center

Computations can be divided among computers in a data center.



Overview - Coordinating Actions in a Network

Network of nodes with communication:



Other work moving information in networks:

- The Gossiping Dons Problem [Bollobas, The Art of Mathematics]
- Distributed Average Consensus [Tsitsiklis, Bertsekas, Athans 84]
- Communication Complexity [Yao 79]
- Function Computation [Ayaso, Shah, & Dahleh 08]

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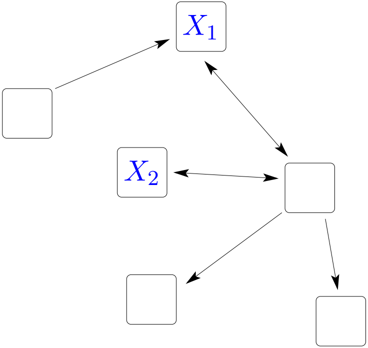

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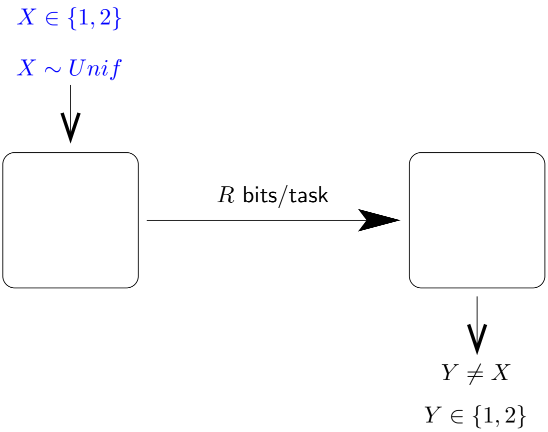
Two Nodes

Tasks are identified by numbers.



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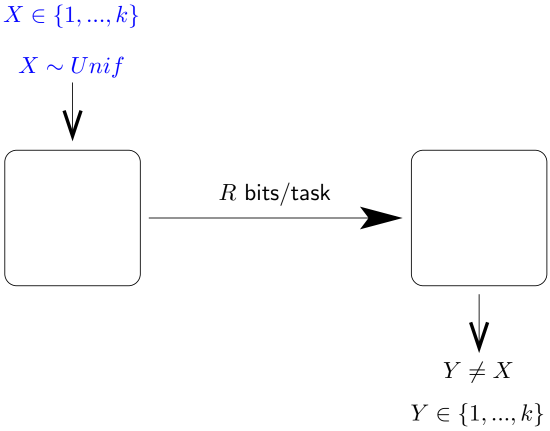

Buffer of Tasks

Example ($R = \frac{1}{4}$, $k = 5$):

Tasks assigned to X: X_1 X_2 ... each independent

Sample realization: 3 1 2 5 3 5 2 4

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Message bits:	$b_1 b_2 = 01$	

Buffer of Tasks

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Tasks assigned to X: X_1 X_2 ... each independent

Sample realization: 3 1 2 5 3 5 2 4

Message bits: $b_1 b_2 = 01$

Codebook:

b_1 b_2	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
0 0	1	2	3	4	5	1	2	3
0 1	2	4	1	3	5	2	4	1
1 0	...							
1 1	...							

Two Nodes

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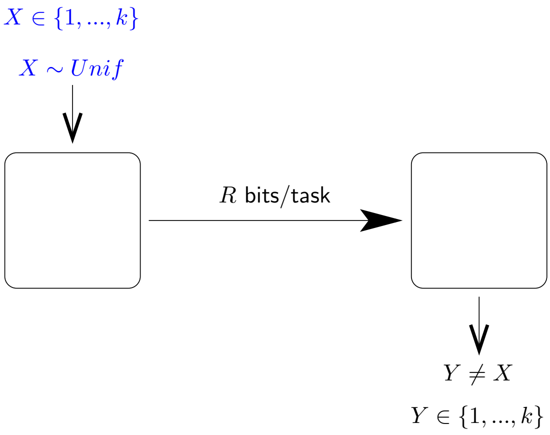

Concept from Rate-distortion Theory

To generate correlated actions $\sim p(x,y)$,

$R \geq I(X;Y)$ is required.

$$
\begin{aligned}
I(X;Y) &= H(X) + H(Y) - H(X,Y), \\
H(X) &= \mathbb{E} \log \frac{1}{p(X)}.
\end{aligned}
$$

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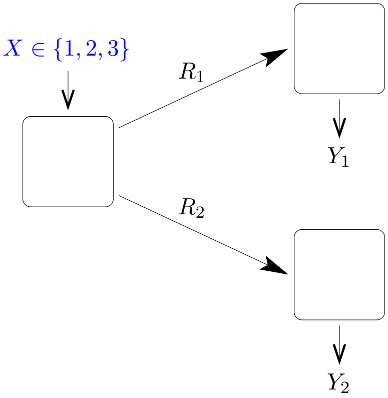
Choose $(X,Y) \sim Unif\{(i,j) \ : \ i \neq j\}$.

$R_{min} = \log\left(\frac{k}{k-1}\right)$.

Three Node Network



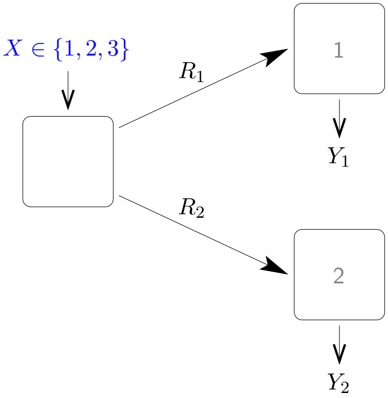
Ideas for assigning tasks uniquely (i.e. $X \neq Y_1 \neq Y_2 \neq X$).

Assign Y_1 first: $R_1 = \log\left(\frac{3}{2}\right)$.

Assign Y_2 using full rate: $R_2 = \log 3$.

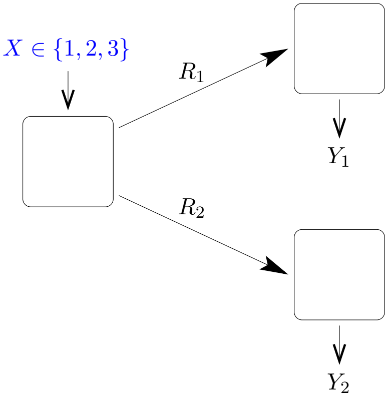
$R_{ave} = \log_2 3 - 1/2$.

Three Node Network



Ideas for assigning tasks uniquely (i.e. $X \neq Y_1 \neq Y_2 \neq X$).
Assign $Y_1 \in \{1, 3\}$: $R_1 = H\left(\frac{1}{3}\right)$.
Assign $Y_2 \in \{2, 3\}$: $R_2 = H\left(\frac{1}{3}\right)$.
$R_{ave} = \log_2 3 - 2/3$.

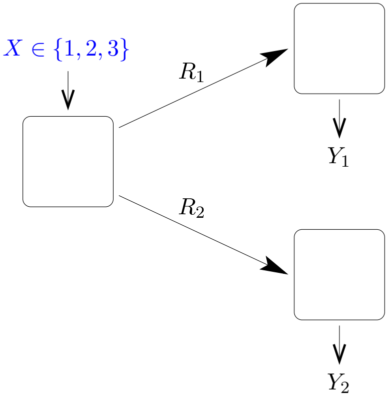
Three Node Network



Techniques so far:

$$
\begin{aligned}
R_1 &\geq I(X;Y_1), \\
R_2 &\geq I(X;Y_2), \\
R_1 + R_2 &\geq I(X;Y_1) + I(X,Y_2) + I(Y_1;Y_2|X).
\end{aligned}
$$

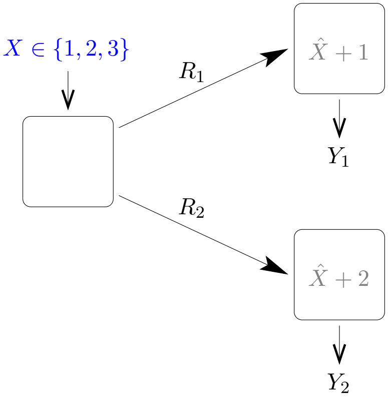
Three Node Network



Common message to both:
(similar to Berger-Zhang scheme for Multiple Descriptions)

$$\begin{aligned} R_1 &\geq I(X;U,Y_1), \\ R_2 &\geq I(X;U,Y_2), \\ R_1 + R_2 &\geq I(X;U,Y_1) + I(X;U,Y_2) + I(Y_1;Y_2|X,U). \end{aligned}$$

Three Node Network



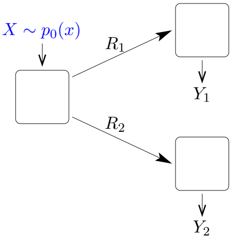
Rates for optimized common message quality $\hat{X}$.

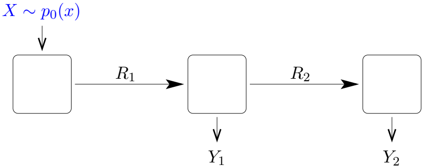
Communication rate with common message: $R_{ave} = \log 3 - \log \phi$.

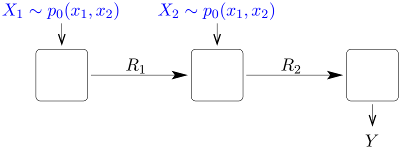
The golden ratio $\phi = \frac{\sqrt{5}+1}{2}$.

[Cuff, Permuter, Cover 09]

Three Node Networks



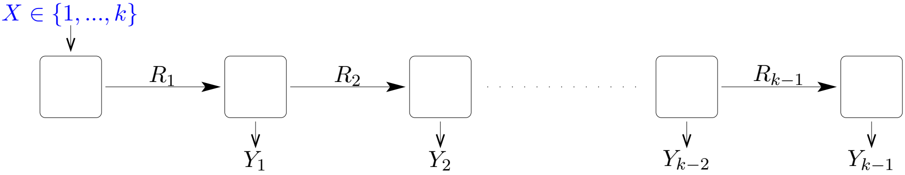




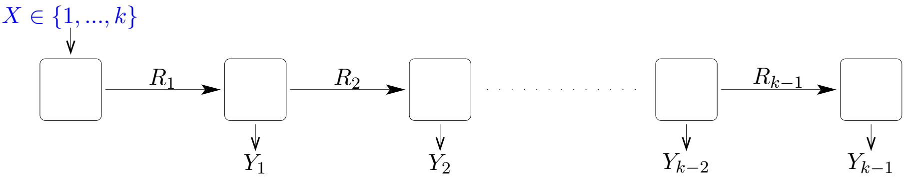
Each of these networks benefits from a common message.

Large Networks

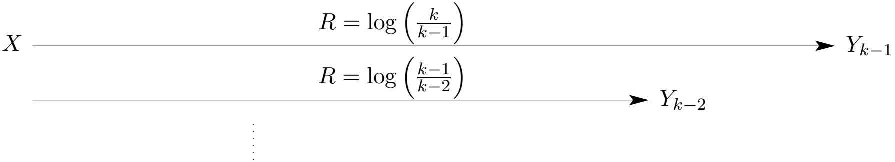
Cascade - One Assigned

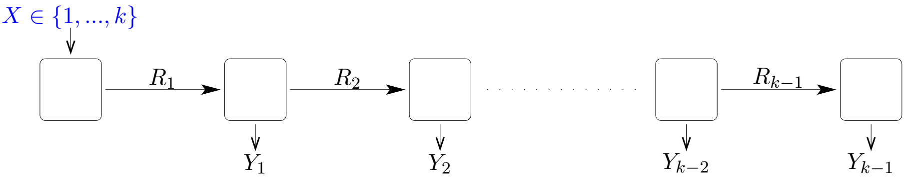


Cascade - One Assigned

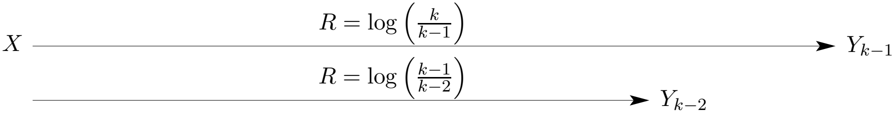


Optimal Communication:



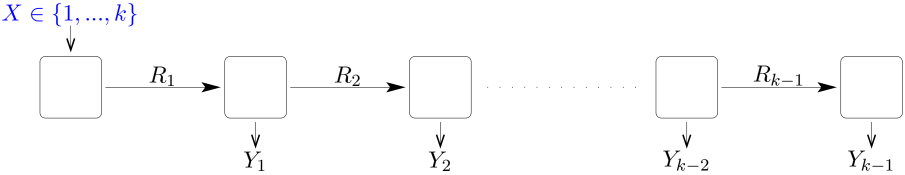


Optimal Communication:



$$
\begin{aligned}
R_{k-1} &= \log\left(\frac{k}{k-1}\right), \\
R_{k-2} &= \log\left(\frac{k}{k-1}\right) + \log\left(\frac{k-1}{k-2}\right) = \log\left(\frac{k}{k-2}\right), \\
R_i &= \log\left(\frac{k}{i}\right).
\end{aligned}
$$

Cascade - One Assigned

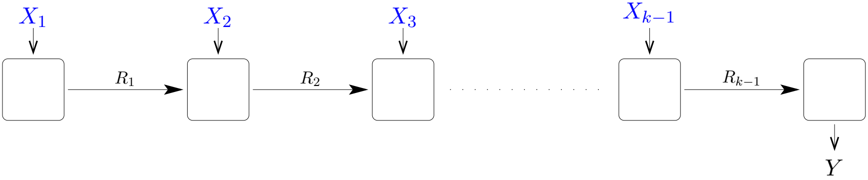


Sum rate:

$$
\begin{aligned}
R &= \sum_{i=1}^{k-1} \log\left(\frac{k}{i}\right) \\
&= k \log k - \sum_{i=1}^{k} \log i \\
&= k \log k - \log k! \\
&\approx k \log k - \log\left(\frac{k}{e}\right)^k \\
&= k \log e.
\end{aligned}
$$

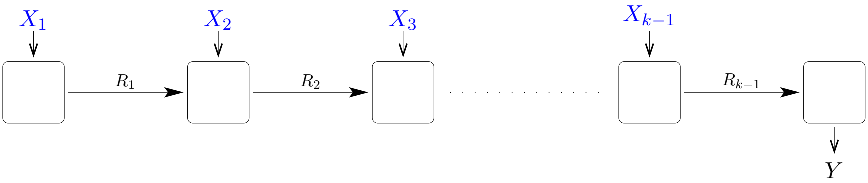
Linear in k

Cascade - All But One Assigned



X_i unique in $\{1, ..., k\}$ for all i.
Y must be the remaining task.

Cascade - All But One Assigned

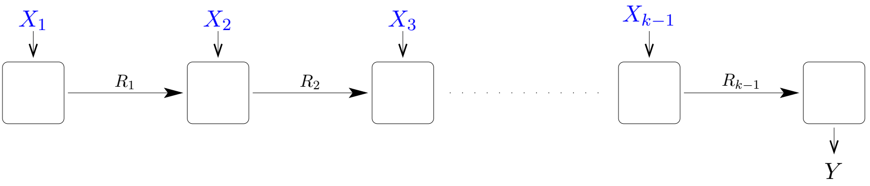


X_i unique in $\{1, ..., k\}$ for all i.
Y must be the remaining task.

Idea - Accumulate information:

$$\begin{aligned} R_1 &= \log(k-1), \\ R_2 &= \log(k-1) + \log(k-2) - \log 2, \\ R_i &= \log \binom{k-1}{i}. \end{aligned}$$

Cascade - All But One Assigned

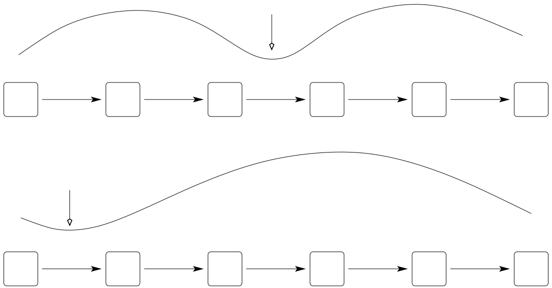


X_i unique in $\{1, ..., k\}$ for all i.
Y must be the remaining task.

Better Idea - Accumulate mod k sum: $\quad R_i < \log k$, for all i.

Sum rate: $\quad R < (k-1)\log k.$

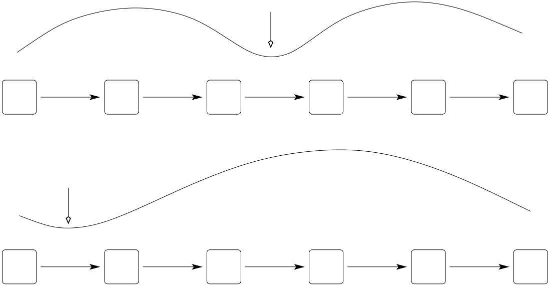
Cascade - Lower Bounds



$R_i \;\geq\; \log(i+1).$

Sum rate: $R \;=\; \sum_{i=1}^{k-1} R_i \;\gtrsim\; k \log \frac{k}{e}.$

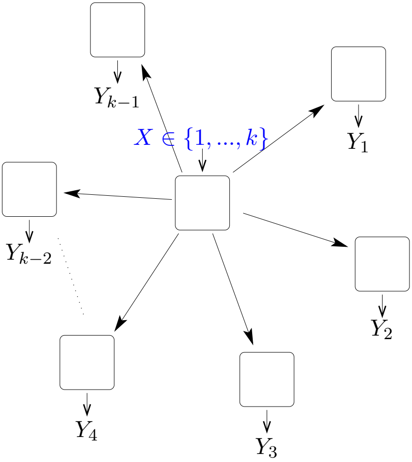
Cascade - Lower Bounds



$R_i \;\; \geq \;\; \log(i+1).$ Sum rate: $R \;\; = \;\; \sum_{i=1}^{k-1} R_i \;\; \geq\approx \;\; k \log \frac{k}{e}.$

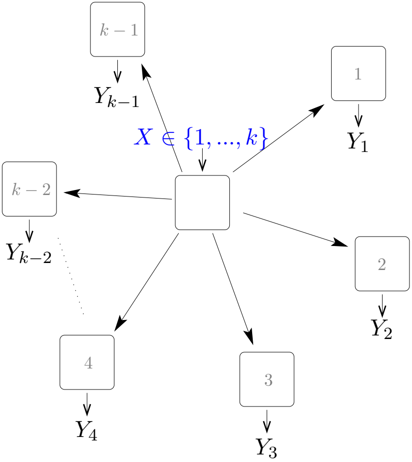
Upper and lower bounds both scale like $k \log k$.

Star Network



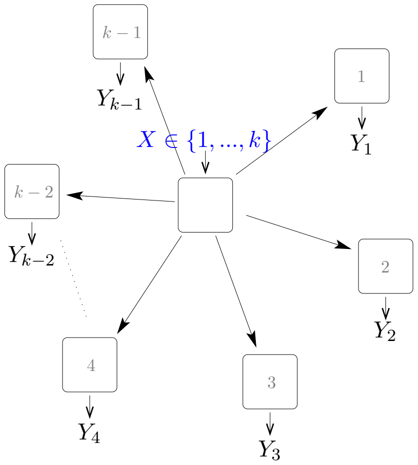
Try $R_i = \log \frac{k}{k-1}$ for all i. (Doesn't work)

Star Network



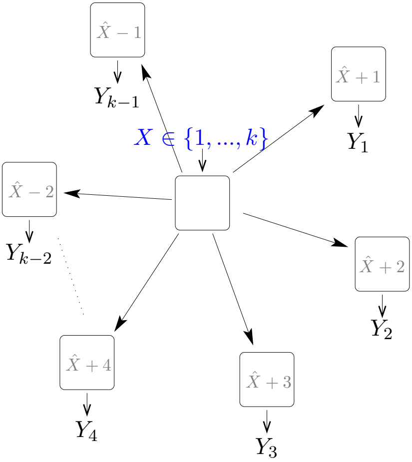
Assign Default Tasks: $R_i = h\left(\frac{1}{k}\right) \approx \frac{\log k}{k}$. Sum rate: $R \approx \log k + \log e$.

Star Network



Lower bound: $R \;\geq\; I(X; Y_1, ..., Y_{k-1}) \;=\; H(X) \;=\; \log k.$

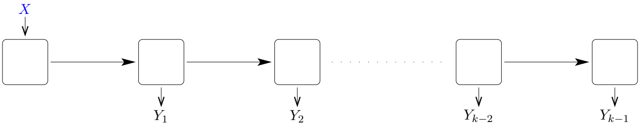
Star Network



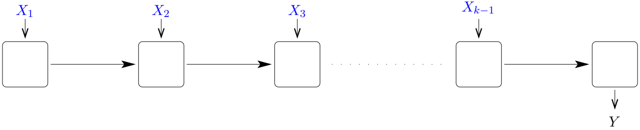
Two phase: Specify low-rate estimate $\hat{X}$. Choose defaults to exclude $\hat{X}$.

Task Assignment Summary

Sum rate:



$R_{min} \approx k \log e$ (linear)



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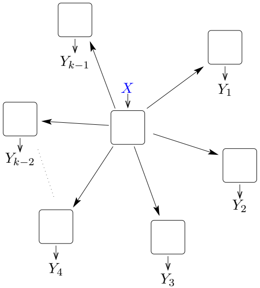

$$R_{min} \approx \log k.$$

[Cuff, Permuter, Cover 09]

$\varsigma\upsilon\kappa\varrho\varepsilon\varsigma\upsilon$

Adversarial Settings

Coordination in the presence of an adversary engages with two frameworks:

1. Cryptography
2. Game Theory

Other work connecting these fields:

[Dodis, Halevi, Rabin 2000]

Game Theory

Payoff Matrix for a zero-sum game:

		Enemy 0	Enemy 1
Me $p(x)$	0	1	2
	1	3	-1

Game Theory

Payoff Matrix for a zero-sum game:

			Enemy	
			0	1
My team $p(x,y)$	00		1	2
	01		3	-1
	10		0	1
	11		-1	0

Team Action

Isolated Participants:

$$p(x)p(y)$$

Team Action

Isolated Participants:

$$p(x)p(y)$$

With Communication:

$$p(x, y)$$

To generate correlated actions $\sim p(x, y)$,

$R \geq I(X;Y)$ is required.

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\begin{aligned}
I(X;Y) &= H(X) + H(Y) - H(X,Y), \\
H(X) &= \mathbb{E} \log \frac{1}{p(X)}.
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This does not produce independent actions in the sequence.

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H(X) &= \mathbb{E} \log \frac{1}{p(X)}.
\end{aligned}
$$

This does not produce independent actions in the sequence.

What is the price of independence?

Billy and the Bully

Billy hopes to avoid the Bully:

Billy, $P(\text{home}) = 3/4$ \ Bully	out	home
home	1	0
out	0	3

Billy and the Bully

Billy hopes to avoid the Bully:

Bully

Billy

$P(\text{home}) = 3/4$

	out	home
home	1	0
0	0	3
1	0	0

If friends go to hangout 0, Billy gets no enjoyment by going to 1.
How much information about hangout choice do friends need to give?

Erasure Challenge

Person A

0 1 0 0 1 1 1 1

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

Erasure Challenge

Person A	Person B
0 1 0 0 1 1 1 1	0 e e e e 1 e e

How much must Person A tell Person B?

Erasure Challenge

Person A								Person B							
0	1	0	0	1	1	1	1	0	e	e	e	e	1	e	e

How much must Person A tell Person B?

- Tell all the bits
  8 bits

Erasure Challenge

Person A

0 1 0 0 1 1 1 1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits

Erasure Challenge

Person A

0 1 0 0 1 1 1 1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits

Erasure Challenge

Person A

0 1	0 0	1 1	1 1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization

Erasure Challenge

Person A							
0	1	0	0	1	1	1	1

Person B

0 e e e e 1 e e

How much must Person A tell Person B?

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Erasure Challenge

Person A							
0	1	0	0	1	1	1	1

Person B							
0	1	e	e	1	1	e	e

How much must Person A tell Person B?

- Tell all the bits
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How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization
  $\log_2 \binom{4}{2} + 4$ bits

Erasure Challenge

Person A

0	1	0	0	1	1	1	1

Person B

0	e	e	e	e	1	e	e

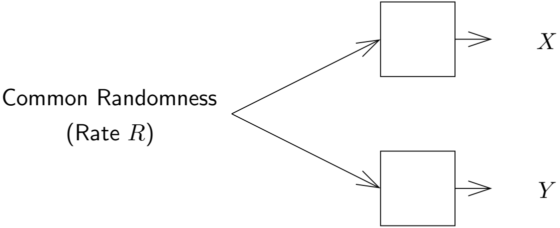
How much must Person A tell Person B?

- Tell all the bits
  8 bits
- Choose the sequence for B and tell it
  $\log_2 \binom{8}{2} + 2$ bits $= \log_2 112 = 6.81$ bits
- Split the randomization
  $\log_2 \binom{4}{2} + 4$ bits $= \log_2 96 = 6.58$ bits

Wyner's Common Information

[Wyner 75]: $$C(X;Y) \triangleq \min_{X-U-Y} I(X,Y;U).$$

Amount of common randomness needed to generate X and Y?



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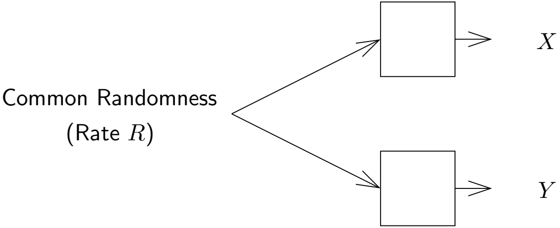
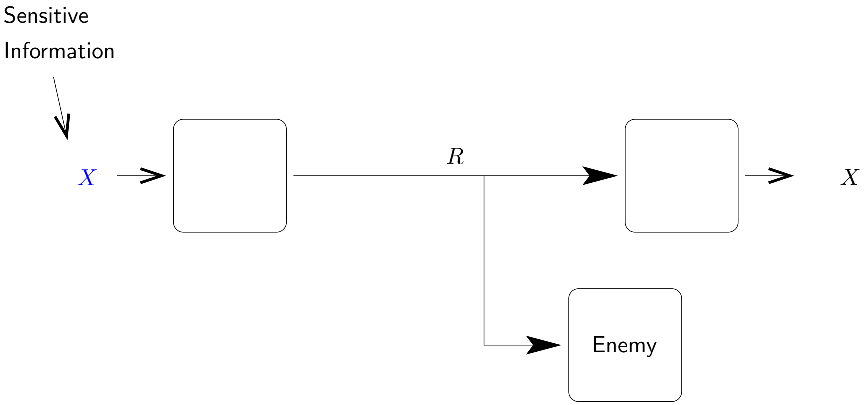
[Wyner 75]: $$C(X;Y) \triangleq \min_{X-U-Y} I(X,Y;U).$$

Amount of common randomness needed to generate X and Y?

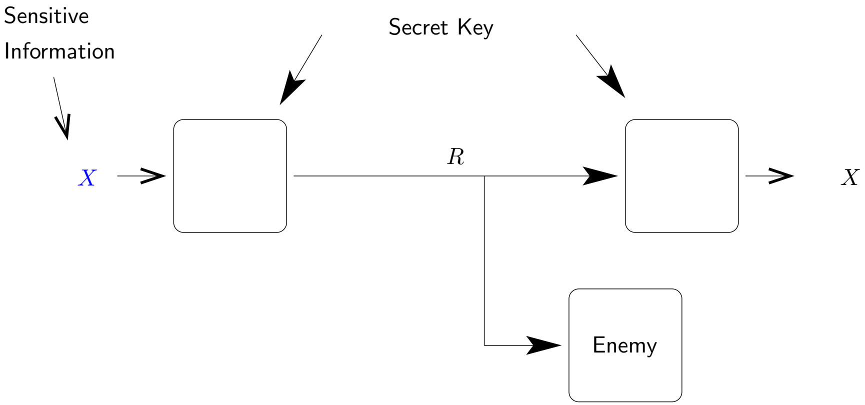


[Cuff 08]: Best use of common randomness in repeated zero-sum game.

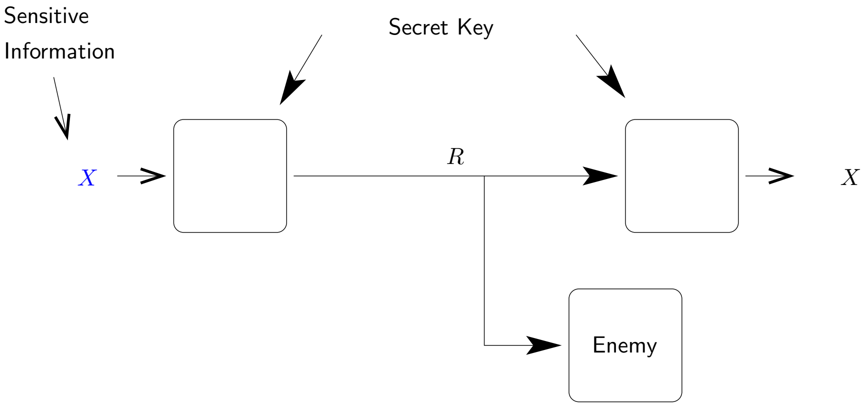
Encryption



Encryption



Encryption



$$R_1 = R_2 = H(X).$$ [Shannon 45]

One-time Pad

Message:

01011011101

Secret Key (random):

11100101101

One-time Pad

Message:

$$01011011101$$

Secret Key (random):

$$11100101101$$

$\bigoplus$

Transmission:

$$10111110000$$

One-time Pad

Message:

01011011101

Secret Key (random):

11100101101

$\oplus$

Transmission:

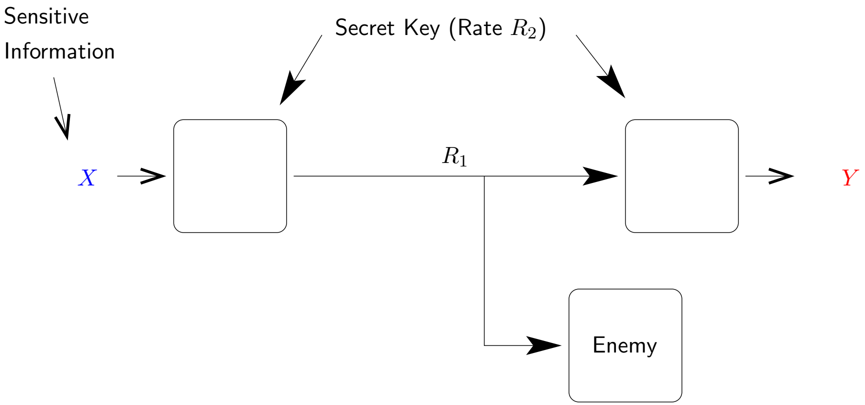
10111110000

$\oplus$

Decoded Message:

01011011101

Correlation Encryption

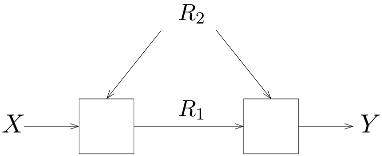


Goals:

1. Y correlated with X according to desired $p(y|x)$.
2. Enemy knows nothing about X or Y.

Correlation Encryption Rate Region

$$S \triangleq Cl\{\text{ encryption achievable } (R_1, R_2)\}$$



Theorem: Encryption Rate Region

$S = \{(R_1, R_2)$:

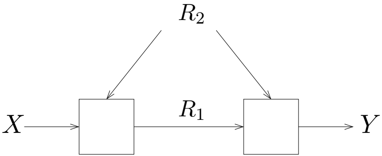
$$\begin{aligned} R_1 &\geq I(X;U), \\ R_2 &\geq I(X,Y;U), \end{aligned}$$

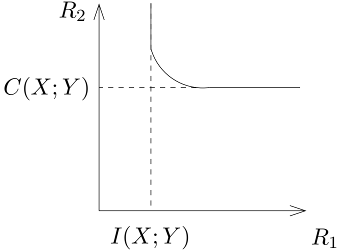
for some U such that $X - U - Y$ forms a Markov chain and $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{Y}| + 1.\}$

[Cuff 08]

Correlation Encryption Rate Region

$$S \triangleq Cl\{\text{ encryption achievable } (R_1, R_2)\}$$





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[Cuff 08]

Achievability

Achievability

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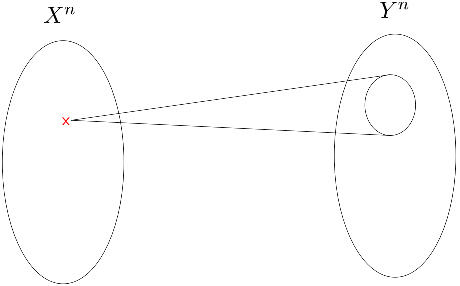
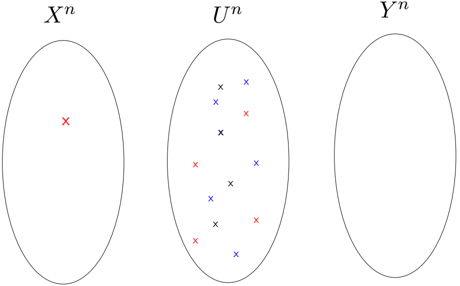
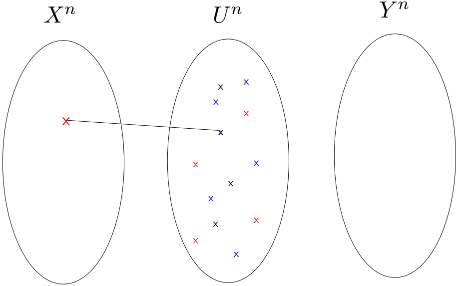


Achievability



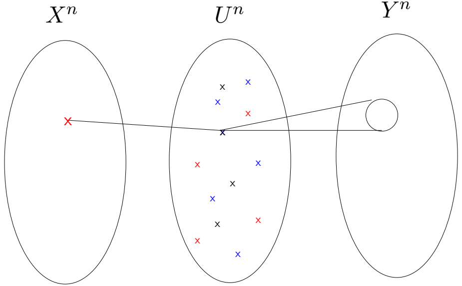
- Generate many codebooks of u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.
- The secret key specifies the codebook to use.

Achievability



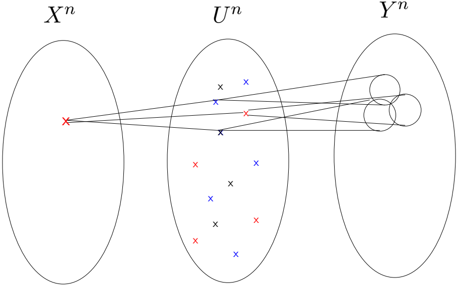
- Generate many codebooks of u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.
- The secret key specifies the codebook to use.
- Encoder finds a u^n sequence correlated with x^n and sends the index i.

Achievability



- Generate many codebooks of u^n sequences $\sim \prod_{i=1}^{n} p(u_i)$.
- The secret key specifies the codebook to use.
- Encoder finds a u^n sequence correlated with x^n and sends the index i.
- Decoder generates y^n randomly conditioned on $u^n(i)$.

Achievability



$$
\begin{aligned}
R_1 &\geq I(X;U), \\
R_2 &\geq I(X;U) + {\color{red} I(U;Y|X)}.
\end{aligned}
$$

Resolvability: [Wyner 75] [Han, Verdú 93]

Converse

M is the message.
W is the secret key.

$$
\begin{aligned}
nR_1 &\geq H(M) \\
&\geq H(M|W) \\
&\geq I(X^n; M|W) \\
&\geq I(X^n; M, W) \ldots
\end{aligned}
$$

Converse

M is the message.
W is the secret key.

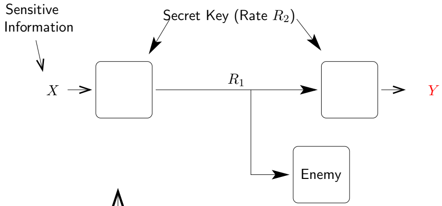
$$\begin{aligned} nR_1 &\geq H(M) \\ &\geq H(M|W) \\ &\geq I(X^n;M|W) \\ &\geq I(X^n;M,W)... \end{aligned}$$

$$\begin{aligned} nR_2 &= H(W) \\ &\geq H(W|M) \\ &\geq H(X^n,Y^n;W|M) \\ &= I(X^n,Y^n;W,M) - I(X^n,Y^n;M)... \end{aligned}$$

Example

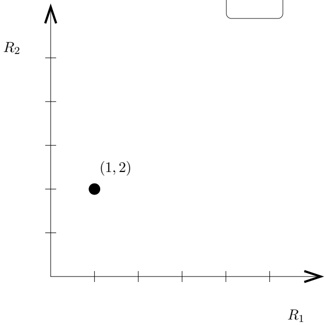
Task assignment in an adversarial setting.

System Monitor



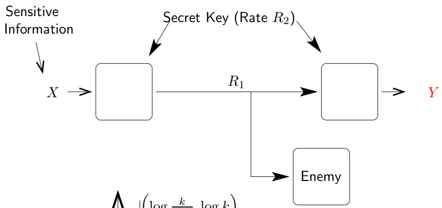
$X \sim Unif\{1, ..., k\}$.

Y needs to be different from X and **random among the choices**.



Task assignment in an adversarial setting.

System Monitor



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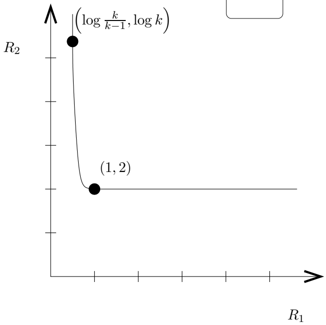
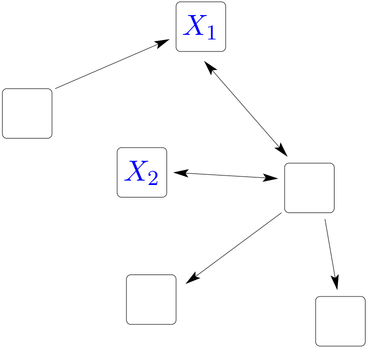


Recap

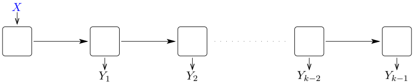
Network of nodes with communication:

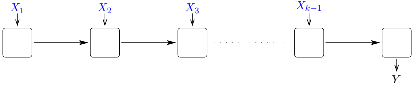


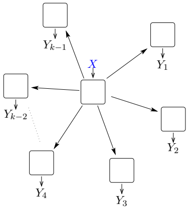
Observations:

- Tools: Random coding, auxiliary variables, common randomness.
- Different networks require very different techniques.

Non-adversarial:







Adversarial:

Two Nodes:

- Secret key required
- Tradeoff between communication and secret key
- Game theory perspective

Fundamental Limits:

- Communication: $R_1 > I(X;Y)$.
- Secret key: $R_2 > C(X;Y)$.

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

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Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

Acknowledgments

M W V P U E C R L I A P O B N H D S A G V S Y A R K E M I O A S R Y N L A W

And many more!

Acknowledgments

S J D Q P O E W N Z M X C N Z H B A H G J A K S D F X P I U C I O P U Q R E
T Y Q W J R H X C M N B C V H A V B C N Z M X M Q P W E O I R E Y W O Z X
C V N B C M L A S D F K G J H Y S T E U Q S J D Q P O E W N Z M X C N Z J D
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L A S D F K G J H Y S T E U Q S J D Q P O E W N Z M X C N Z