Condorcet Voting Methods Avoid the Paradoxes of Voting Theory

(Invited Paper)

Tiance Wang
Princeton University
Princeton, New Jersey 08544
tiancew@princeton.edu

John Sturm
Harvard College
Cambridge, MA 02138
jsturm@college.harvard.edu

Paul Cuff
Princeton University
Princeton, New Jersey 08544
cuff@princeton.edu

Sanjeev Kulkarni
Princeton University
Princeton, New Jersey 08544
kulkarni@princeton.edu

***Abstract*—Democratically choosing a single preference from more than two candidate options is not a straightforward matter. In fact, voting theory has established a number of paradoxes which assert seemingly innocuous attributes to be incompatible. One of the most desirable attributes—independence of irrelevant alternatives—is proven by Arrow to be incompatible (in a worst-case sense) with nominal fairness constraint. Another theorem states that all voting systems will have opportunities for a voter to improve their outcome by voting contrary to their true preferences.**

What we show in this work is that Condorcet methods, which uniquely satisfy the independent of irrelevant alternatives property whenever possible, actually avoids these paradoxes both in practice (based on real data) and in theory (in a probabilistic sense).

(This is an extended abstract for the talk presented at the Allerton Conference.)

I. Introduction

Arrow's impossibility theorem [1] and the Gibbard-Satterthwaite theorem [2] [3] assert that any single-winner deterministic voting system inevitably runs into two paradoxes: if it treats all voters equally and all candidates equally (as it should for a democratic voting system), then it fails the Independence of Irrelevant Alternative (IIA) property and is susceptible to strategic voting. For any voting system, there are some voting profiles that fail these properties, where *voting profile* means the collection of all votes. However, these results focus on the "worst case" rather than the usual case. The question "how often does a voting system violate these properties" remains an open question.

Consider an election with N voters and M candidates. Assume the voting system is anonymous (treats all voters equally) and neutral (treat all candidates equally). Each voter expresses their opinion with a complete ranking, or full linear ordering of all M candidates. We further assume that the voting system must be a majority rule when there are only two candidates.

The *Condorcet winner* is the candidate who beats all other candidates in pairwise matches. In other words, the Condorcet winner is preferred to any other candidate by more than half of the population. A Condorcet winner may or may not exist for a voting profile. A Condorcet method is any voting system that always elects the Condorcet winner when one exists.

We make three arguments on how Condorcet methods excel in avoiding the two paradoxes of voting theory, from the perspective of logic, data and probabilistic analysis. In this short summary, we first present all three arguments, then focus on the third argument, i.e. Condorcet methods are more robust to strategic voting than other methods.

II. Three Arguments for Condorcet Methods

A. First Argument

First argument: Condorcet methods satisfy the IIA property whenever possible. Suppose a voting profile satisfies IIA. By the definition of IIA, getting rid of a losing candidate does not change the winner. Now if we get rid of all but one losing candidates, then the winner must beat this remaining candidate. Therefore, the winner has to be a Condorcet winner. In other words, if we want to satisfy IIA for as many voting profiles as possible, then a Condorcet method is the optimal choice. On the other hand, if a Condorcet winner does not exist, then there is no hope to satisfy IIA, as there must be three candidates a,b and c forming a loop, i.e. a beats b, b beats c and c beats a. Such voting profiles lead to a *Condorcet paradox*.

B. Second Argument

It is natural to ask whether a Condorcet winner is likely to exist in practice. We analyze this question with some survey results and some online election data. In our 2012 US presidential survey, we advertised online and collected 1,415 responses on people's preferences for presidents. We asked each respondent to evaluate 11 choices for president, including 7 Republicans, 2 Democrats and 2 Libertarians. The aggregated results show that a Condorcet winner always exists for any subgroup of candidates.

There are publicly available election data online [4][5] with similar results. In the Tideman Collection [4], Condorcet winners exist in 75 out of 78 elections. In the Debian Project Voting Data [5], Condorcet winners always exist in 18 polling results.

C. Third Argument

We show how Kemeny-Young method, and Condorcet methods in general, are more robust to strategic voting than other methods.

Recall our election setting: there are N voters and M candidates, and each vote is a complete ranking of all candidates. The set of all voting profiles is a simplex in an $M!$-dimensional space. When the number of voters goes to infinity, we can replace it with a probability simplex of the same dimension. A voting system then is a partition of the simplex into different cells, each corresponding to a winning candidate. Strategic voting only happens if the voting profile lands right on a boundary or very close to a boundary between two cells.

Some voting systems only depend on pairwise comparison results. That is, to determine the winner, we only need to know the portion of the population that prefer any candidate to any other candidate. Condorcet methods and Borda Count are such voting systems. When there are only $M = 3$ candidates, the dimension of the input can be reduced from 6 to 3, making it much easier to visualize the voting systems.

1) Strategic Voting for Borda Count: Borda Count ranks candidates by summing up the points received from all voters. Each voter contributes 2 points to their top choice, 1 point to their second choice and no point to their bottom choice. Therefore, the simplex is partitioned according to the highest scorer. Consider an example:

Example 1. In an election using Borda Count, the scores of three candidates a, b, c are $S_a = 100, S_b = 100$ and $S_c = 70$. A voter with preference cba, if somehow has access to the entire voting profile, is willing to *compromise* and vote bca instead of cba in order to help her second choice win. Another voter with preference abc, knowing that b threatens her top choice a, will *bury* b by voting acb. In both cases, two voters could benefit from voting not according to their true preference.

By careful inspection, we find that there are always groups of strategic voters if the two top candidates are tied or only 1 point apart. Figure 1 shows how the boundaries of Borda Count partitions the simplex. The axis U_{ab} denote the portion of the population that prefer a to b, normalized to $[-1, 1]$. That is, $U_{ab} = 0$ means there are exactly one half of the population who prefer a to b. U_{bc} and U_{ca} are defined similarly. The light-colored polehydron in the background is the space of all voting profiles.

2) Strategic Voting for the Kemeny-Young method: The Kemeny-Young method assigns a score to each ranking, called the Kemeny score. The Kemeny score of a ranking is calculated by summing up the Kendall tau distance between that ranking and each vote. The ranking with the lowest Kemeny score is the optimal ranking, and the winning candidate is the first place of the optimal ranking. Thus for the Kemeny-Young method, the cells are formed by first partitioning the space into 6 cells according to the lowest Kemeny score, then merge two cells sharing the same winner.

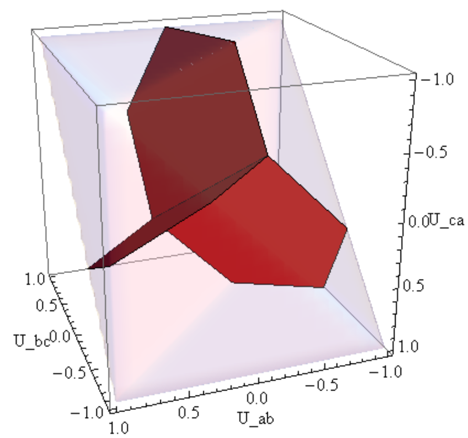

Fig. 1. Boundaries for the Borda Count

On investigating different voting systems we discovered an interesting fact: For the Kemeny-Young method, there are two types of boundaries: the ones between two rankings with Kendall tau distance 1, such as abc and bac; and the ones between two rankings with Kendall distance 2, such as abc and cab. Boundaries of the first kind is completely immune to strategic voting! In other words, if a voting profile lands on or near this boundary, there exist *pivotal voters*, who can change the voting outcome by changing their own votes. However, they are not willing to do so because the new winner would be a less preferred candidate. For this reason we call them "non-strategic boundaries". Strategic voting only happens when the voting profile is on or near the "strategic boundaries".

Furthermore, the non-strategic boundaries do not intersect with the set of profiles that lead to the Condorcet paradox. Therefore, all Condorcet methods share the same non-strategic boundaries. and the strategy-proof property of the Kemeny-Young method applies to all Condorcet methods in general. The second argument that a Condorcet winner is likely to exist in practice further strengthens this result. Figure 2 shows the strategic boundaries (triangular) and non-strategic boundaries (squares), with the entire set of voting profiles in light color.

3) Quantitative Analysis: To motivate a quantitative analysis of the probability of strategic voting, we need a probabilistic model for voting. Assume the voter randomly picks a vote according to a *social bias* p. The voting profile V is thus subject to a multinomial distribution: $V \sim \text{multi}(N, p)$. We do not place any restriction on the bias p. If p is closer (in the sense of KL-divergence) to a non-strategic boundary than a strategic boundary, then the probability that a voter is strategic conditioned that he is pivotal asymptotically goes to 0 as the population goes to infinity, a result of the Sanov's theorem [6]. We call this probability the *conditional incentive*. However, non-Condorcet methods such as Borda

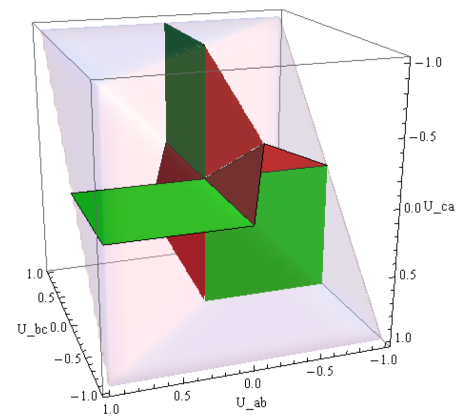

Fig. 2. Strategic boundaries (triangular) and non-strategic boundaries (squared) for the Kemeny-Young method.

Count, plurality and instant run-off voting only have strategic boundaries. Therefore, for any social bias, the conditional incentive does not converge to 0.

With the multinomial distribution assumption, the conditional incentive can be calculated precisely. It only depends on the social bias and the voting system used. It should be clarified that this probability model is for no more than analytical purpose. With any other model of vote distribution, the distinction between strategic and non-strategic still exists, and the distance metric is probably similar, though not identical to the KL-divergence that we use.

p (social bias)	$I_c(p,N)$	$I_c(p)$	error
[.2 .2 .18 .18 .12 .12]	.2474	.2453	.85 %
[.2 .15 .15 .15 .2 .15]	.2597	.2605	.32 %
[.18 .18 .17 .17 .15 .15]	.2636	.2595	-1.58 %
[.2 .2 .1 .1 .2 .2]	.2371	.2399	1.17%
[.19 .18 .17 .16 .15 .15]	.2628	.2599	-1.11%
[.2 .18 .17 .16 .15 .14]	.2585	.2556	-1.12%
[.2 .15 .18 .17 .15 .15]	.2618	.2521	-3.71%

TABLE I
CONDITIONAL INCENTIVE FOR BORDA COUNT WITH $10,000$ VOTERS

p (social bias)	$I_c(p,N)$	$I_c(p)$	error
[.2 .2 .18 .18 .12 .12]	.1598	.1631	2.05 %
[.2 .15 .15 .15 .2 .15]	.2010	.1983	-1.31 %
[.18 .18 .17 .17 .15 .15]	.2005	.1988	-.85 %
[.2 .2 .1 .1 .2 .2]	.1335	.1342	.50%
[.19 .18 .17 .16 .15 .15]	.2006	.1993	-.64%
[.2 .18 .17 .16 .15 .14]	.1941	.1933	-.39%
[.2 .15 .18 .17 .15 .15]	.2007	.2000	-.35%

TABLE II
CONDITIONAL INCENTIVE FOR PLURALITY WITH $10,000$ VOTERS

p (social bias)	$I_c(p,N)$	$I_c(p)$	error
[.1667 .1652 .2036 .1147 .2377 .1120]	.2080	.1880	-9.6%
[.3241 .0345 .2603 .1794 .1299 .0718]	.1092	.1079	-1.2%
[.1797 .2276 .2498 .0081 .2648 .0699]	.3081	.3281	6.1 %
[.1171 .3045 .3582 .0400 .0712 .1090]	0	0	0
[.2503 .2588 .0954 .2088 .0744 .1123]	0	0	0
[.0811 .2181 .1935 .1630 .1144 .2299]	.2784	.2818	1.2 %
[.2226 .0980 .2792 .0342 .3548 .0112]	.1730	.1721	-0.5 %

TABLE III
CONDITIONAL INCENTIVE FOR KEMENY-YOUNG WITH $10,000$ VOTERS

Due to space constraints, we only show a few social biases with the analytical ($I_c(p)$ column) as well as the simulation values ($I_c(p,N)$ column) of the conditional incentive. The six probability masses are listed in the order of $abc, acb, cab, cba, bca, bac$.

For the Kemeny-Young method, we use some randomly generated biases as all but one of the above biases give a conditional incentive of 0.

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