A Behavioral Model of Bargaining with Endogenous Types

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Current Version: November 2003

Abstract

We enrich a simple two-person bargaining model by introducing "behavioral types" who concede more slowly than does the average person in the economy. The presence of behavioral types profoundly influences the choices of optimizing types. In equilibrium, concessions are calculated to induce "reciprocity": a substantial concession by player i is followed by a period in which j is much more likely to make a concession than usual. This favors concessions by i that are neither very small nor large enough to end the bargaining immediately. A key difference from the traditional method of perturbing a game is that the actions of our behavioral types are not specified in absolute terms, but relative to the norm in the population. Thus their behavior is determined endogenously as part of a social equilibrium.

*We would like to thank Faruk Gul, Ashvin Rajan, Ennio Stacchetti, and Tim VanZandt for very helpful comments. We thank Alvaro Bustos, Sofronis Clerides, and Sebastian Ludmer for their assistance. We are indebted to Yuliy Sannikov for exceptionally skillful research assistance. The authors gratefully acknowledge the support of the National Science Foundation and the Russell Sage Fundation.

1 Introduction

Strategic posturing appears to be an important part of many bargaining situations. Each party would like to be perceived as someone who is unlikely to yield ground easily. Ironically, the models of Rubinstein (1982) and Stahl (1972) that transformed economists' views of bargaining have such strong implications that they leave no room for players to build reputations for being hard bargainers. Shares of the surplus to be divided are determined completely by individuals' valuations and time preferences, a particularly remarkable fact in Rubinstein's infinite horizon setting. There are no delays before settlement is attained; players understand what their equilibrium shares are, and agree to them immediately.

Richer models are required to explain the costly negotiations (strikes during labor disputes being the most obvious example) that we frequently observe in reality. An extensive literature[1] investigates the kinds of bargaining and inefficiencies that can result from the value of a player's discount factor (or other taste or technological parameter) being unknown to her opponent. For example, player A may hold out for a considerable time for a favorable settlement, hoping to convince player B that A has a low rate of time preference (and consequently, a strong bargaining position).

Even in settings in which most relevant information about the bargaining environment is common knowledge, each side is often concerned about how stubborn the opponent may be, and about how her own strategic posture is perceived by that opponent. Abreu and Gul (2000) create a role for such considerations by introducing at least a small possibility that each player might be a compulsive type who will *never* settle for less than, say, two-thirds of the surplus. If you can convince your opponent that you are such a type, it is rational for her to acquiesce to that demand right away. Thus, there are incentives for rational players to imitate the compulsive types, and this dominates the nature of the strategic equilibrium.

Perturbing a dynamic game by introducing "crazy types" with some small probability is a device pioneered by Kreps and Wilson (1982) and Milgrom and Roberts (1982). It has powerful consequences for reputation formation in various contexts, including entry deterrence, Stackelberg leadership, and self-enforcing cooperative agreements (see Fudenberg (1992) for an authoritative survey). In the bargaining problem, the particular perturbation chosen by Abreu and Gul yields both equilibrium uniqueness and robustness to the fine details of how the game is specified (such as the exact timing of offers and counteroffers), while still allowing a major role for strategic posturing.

Along with its many attractions, the "perturbing with crazy types" methodology has some drawbacks. Manipulating the kind of perturbation introduced can have a big impact on what equilibria look like, a point made dramatically

[1] See for instance, Chatterjee and Samuelson (1987), Cho (1990), Cramton (1984), Fudenberg, Levine, and Tirole (1985), Gul and Sonnenschein (1988) and Hendricks, Weiss, and Wilson (1988).

by Fudenberg, Kreps and Levine (1988). Further, in some cases the behavior of rational players simply mimics that of the crazy types in a naive way. For example, in Abreu and Gul (2000) a rational player always makes exactly the same demand each time she makes an offer; there is no progress made in bargaining until someone gives in completely to the other person's monotonous demand.

We propose a different way of perturbing a game to permit strategic posturing. The "types" introduced are defined by how their behavior differs from the average behavior an observer sees in equilibrium. Think of one of these types as having a behavioral *bias* of a particular kind. This could involve making smaller concessions than people typically make in each circumstance, or being less likely to make any given concession, or waiting longer between improvements in offers, and so on. One can interpret such a type's decisions as follows: she is guided by commonly observed features of bargaining in her community or society, but is subject to a behavioral bias that systematically distorts her own choices in some direction. More generally, beyond the sphere of bargaining, all of us encounter behavioral biases in different acquaintances: some are unrealistically sure of themselves, some overly demanding (while thinking of themselves as reasonable), some unjustifiably pessimistic, and so on. Experimental psychology offers rich support for the prevalence of a variety of cognitive and behavioral biases.[2]

Operationally, what we are suggesting here differs from a traditional perturbation type in that the strategies of our behavioral types are not known until the social equilibrium (which includes them) is calculated. Whereas a traditional perturbation type's actions in each circumstance are specified exogenously, the behavioral type's action in any contingency is *endogenous*, determined by the logic of equilibrium. To an unboundedly rational modeler, the distinction would be of little significance: she would instantly see the implications of the behavioral bias she specified (in relative terms) for the equilibrium strategy of the perturbation type she might introduce, and therefore she could just as well introduce that strategy as an exogenous perturbation in the first place. [3]

Not all modelers (or even all economists) are unboundedly rational, however. Many of us prefer to let information and incentives play as full a role as possible in shaping the predictions of our model. So it is more revealing to see a subtle pattern of strategic behavior (for both perturbation and rational types) emerge from some simple behavioral bias, than to have rational behavior merely mirror the exogenous behavior imposed on perturbation types by the modeler.

It is also more difficult, generally speaking, to work with endogenous perturbations. All types' equilibrium strategies are determined simultaneously in a fixed point calculation, and even the most innocent-looking behavioral bias can lead to complex contingent plans for each type. In this paper we choose the most

[2] For an excellent survey see Rabin (1997).

[3] The game with "endogenous types" might have multiple equilibria, but whichever one interested our modeler could be replicated in a game with exogenous perturbations.

tractable behavioral bias that captures the idea that some people are less likely to make a given concession than others, after any particular negotiation history. The implications of introducing just one of the endogenous types (for each player) are worked out in a modest example in which two bargainers are dividing five homogeneous units of a good.

Section 2 presents the example, and Section 3 develops closed-form solutions for subgames in which the bargainers are only one unit away from agreement (their demands sum to six). Sections 4 and 5 use these solutions to study equilibrium in the remaining subgames.

Our interest in this example lies in determining the essential qualitative features of how bargaining proceeds. The "one unit from agreement" subgames look very much like the Abreu and Gul problem: they have unique solutions taking the form of wars of attrition (each side hoping the other will give in soon).

While this waiting game proceeds, player i is indifferent between holding out for a concession or giving up. Hence, her expected continuation payoff is just what she has been offered by j, that is, the residual remaining after j's demand has been met. Interestingly, during wars of attrition in subgames where players are more than one unit away from agreement, expected payoffs do *not* always have this residual property. When the sum of players' demands exceeds the available five units by three, for example, each player may (depending on their reputations) be randomizing between conceding two units, and not conceding at all. Each of these choices may be *strictly* preferable to conceding all three contested units. If so, when i makes a two-unit concession, j immediately makes a *reciprocal* concession (of the remaining unit) with positive probability. Sections 3 and 4 explain reciprocity in some detail.

The literature contains other examples of one concession provoking another via quite different mechanisms. Fudenberg and Kreps (1987) consider an incomplete information war of attrition involving one multi-market firm facing independent competitors in each market. In the two market case, for example, if at some moment the competitor in market 1 concedes to the multi-market firm, the competitor in market 2 responds by giving in immediately as well. Whinston (1988) studies multi-plant oligopolists sequentially exiting a shrinking market. One firm may keep two plants open until a competitor (irreversibly) closes a plant, at which point the first firm promptly closes one of its own plants (to save costs and enjoy higher prices).

In our example, it turns out that in some circumstances, a one-unit concession will not be enough to induce reciprocity, but a two-unit concession will. A rough theory of optimal concession size emerges: player i should concede enough to enjoy a reciprocal response, but not so much that the other side has little or nothing left to concede in return. In our example, in many instances there is exactly one size of concession a player is willing to make, following a particular history. The timing of concessions is stochastic; equilibrium determines a unique distribution of concession times.

Our example is just complicated enough to establish that large, infrequent concessions may serve a bargainer better than would a greater number of concessions of minimal size. In our model, one often observes bargaining terminating as follows: a substantial period of mutual obstinacy in which no progress is made is interrupted by a noticeable improvement in 1's offer, which causes 2 to reduce her demand enough for agreement to be reached. The story is familiar to us from observing houses being sold, strikes being settled and family disputes resolving themselves.

Smith and Stacchetti (2001) examine *complete* information bargaining problems, restricting attention to equilibria with a Markovian flavor. They demonstrate the existence of a rich class of solutions in which players make random partial concessions in iterated wars of attrition. Reciprocity does not arise. Nor does it play a role in Bulow and Klemperer (1999), where concessions are all-or-none, but iterated wars of attrition occur as players drop out one by one.

Section 6 considers bargaining over an arbitrary number of units, allowing for a broader class of behavioral biases. The results here are less concrete, but among other things we identify sufficient conditions for the phenomenon of reciprocity to arise. We also indulge in some speculation about further properties of equilibria in these more general settings. Section 7 concludes.

2 The Model

We consider a two-player, infinite-horizon bargaining game in continuous time. The players, indexed by $i = 1, 2$, must agree on how Q indivisible units of a good are to be allocated between them. Sections 2 and 3 give partial treatments of games with arbitrary values of Q; to get full solutions, sections 4 and 5 limit themselves to the case where $Q = 5$. Section 6 returns to the general case.

The players have initial demands x_i^0, where $Q \geq x_i^0 \geq 1$. The initial demands (which we take to be exogenous[4]) are incompatible, that is, $x_1^0 + x_2^0 > Q$. We assume that Q, x_1^0, and x_2^0 are integers. Bargaining proceeds by the players' making successive integer reductions to their initial demands. The game ends at the first time t at which the players' outstanding demands x_i at t satisfy $x_1 + x_2 \leq Q$. If two simultaneous concessions result in demands x_1 and x_2 that sum to strictly less than Q, an event that happens with probability zero in equilibrium, the unclaimed surplus is awarded randomly: the final division is taken to be $(x_1, Q - x_1)$ or $(Q - x_2, x_2)$ with equal probability.

In equilibrium, player 2 sometimes wishes to respond as soon as possible to a

[4]They might be regarded as having been determined in some larger game that resulted in the game under study; alternatively one may think of $Q - x_0^j$ as the "status-quo" payoff to player i were she to concede completely and immediately to her opponent's demands.

concession by player 1 at time t. In continuous time, this is modeled by allowing for two or more "consecutive" concessions to occur at the same instant, and for player 2 to condition her choice of concession at t on player 1's concession at t. One must nonetheless guard against familiar pitfalls of continuous time modeling, such as the failure of a strategy profile to determine a unique path. These difficulties do not arise in our model because players are allowed to adjust their demands in only one direction, that is, downward. This spares us scenarios such as: at time t, player 1 reduces her demand from 10 to 9. Player 2 responds immediately by reducing his demand from 10 to 9, 1 counter-responds (still at time t!) by increasing her demand to 10 once again, 2 switches back to 10, and so on. With the monotonicity we impose, there could be a flurry of concessions and counter-concessions at t, but it would end in a finite number of steps. The formulation of histories and strategies for the bargaining game given in the Appendix avoids any continuous time pathologies, while affording the players all the flexibility they need.

The two players are essentially engaged in a war of attrition.[5] Each is waiting, at any point in the bargaining game, for the other side to concede either completely (that is, reduce her demand enough to end the game) or partially. Behavioral types are less likely to concede in any given time interval than normal types. Our informal story is that a social tradition has evolved regarding how bargaining proceeds. Everybody knows, for example, how likely it is that player 1 will reduce her demand by 3 units, in some specific time interval after a particular history of bargaining. "Normal," or optimizing types, maximize expected utility taking this tradition as given (the usual equilibrium assumption). Behavioral types have also absorbed the social tradition, and think they are acting in accordance with it, but their behavioral bias leads them to concede with lower probability in each circumstance than the social average would dictate. Since the population is composed of both normal and behavioral types, this bias means that normal types concede more quickly than the population average, while behavioral types concede more slowly than that average. We do not attempt to model the evolution of the social tradition, but limit ourselves to examining its eventual outcome.

We turn now to a more precise description of the game, relegating mathematical details to the Appendix.

Histories and Strategies. Suppose that at time t, player i reduces her demand. We call this a "concession episode." If j responds by reducing her demand immediately (or later), this is another concession episode. Any history of play can be regarded as a sequence of concession episodes, indexed by their times (and order)[6] of occurrence. Before any concessions take place, we say players are

[5] More precisely, we will see that it is an *iterated* war of attrition, involving many "layers" of successive concessions.

[6] Concession times are not always sufficient for determining the order in which concessions occurred, because of the possibility of j's responding immediately to a concession by i.

in the "first round" of the concession game; after $k-1$ concessions, they are in the k^{th} round.

A strategy for a particular type (normal or behavioral) of player i is a collection of "local strategies," one for each round of concession. The local strategy for the normal type of player 1 for the n^{th} round, for example, specifies, for each $n-1$ episode history, a probability measure that governs i's concession behavior after that history. It will turn out that in equilibrium, a player will randomize over her concession time (as one expects in a war of attrition) and sometimes over the amount she concedes.

Behavioral Types. Each player might be an optimizer (a normal type) or a behavioral type. The initial probability that i is behavioral is z_i^0. The superscript emphasizes that player j's beliefs about i's type evolve as play proceeds, starting at z_i^0. We often refer to z_i as i's *reputation*.

Player i's normal type maximizes expected discounted utility using the discount factor $r_i > 0$. If agreement is reached at time t with final allocation (x_1, x_2), player i's utility for the game is $x_i e^{-r_i t}$. If agreement is never reached, utility for both players is zero.

We need to specify the bias of the behavioral types in situations in which concessions are made according to some density function, and alternatively for concessions made with positive probability at a particular time t (after a given history). We call these respective cases "continuous concessions" and "lumpy concessions": these terms do *not* refer to the *amounts* conceded, which must always be discrete. Our formulation is a tractable way of expressing the idea that behavioral types tend to take longer to concede, on average, than optimizing types. Some notation will help make this precise.

In a social equilibrium (a bargaining tradition as described earlier), after a sequence of concession episodes player i's next concession time (integrating over different possible concession amounts and both types of player i) will be governed by some probability distribution which we call $F_i(\cdot)$. That is, $F_i(t)$ is the probability (from j's point of view) that i will lower her demand before or at time t, conditional upon j's not conceding by then.[7] Implicitly, F_i is a function of the history of concession episodes, but we suppress this for notational convenience. The distribution F is a weighted average of the distributions for normal and behavioral types. We denote the latter distributions G_i and H_i respectively. That is,

$$F_i = (1 - z_i) G_i + z_i H_i$$

where z_i is the posterior probability that player i is behavioral, after the sequence of concession episodes in question. When the corresponding density functions are well-defined, they are denoted by f, g and h.

[7] To account for the possibility that concession never occurs, F is defined on the extended (non-negative) reals.

The Behavioral Bias. We require that a behavioral type is less likely to concede than is the rational type of player i, by imposing the following condition relating the concession distribution functions F_i, H_i :

$$1 - H_i(t) = (1 - F_i(t))^{\alpha_i} \tag{1}$$

where $\alpha_i \in (0,1)$ is a parameter.

We motivate the assumption by noting that when the corresponding density functions are well defined (1) implies that the conditional densities satisfy:

$$\frac{h_i(t)}{1 - H_i(t)} = \alpha_i \frac{f_i(t)}{1 - F_i(t)} = \alpha_i \lambda_i(t)$$

where $\lambda_i(t)$ is player i's conditional density (or hazard rate) of concession. That is, the behavioral type imitates the population average with the downward proportional bias $\alpha_i \in (0,1)$.

We disregard degenerate solutions in which one of the players gives up with certainty at time 0, whether she is behavioral or not. Such solutions clearly violate the spirit of the model, which is that behavioral types are less likely to concede than the population average.

In equilibrium, in a subgame that might be reached by a behavioral player but not by normal types, the only strategy consistent with our rule for behavioral players involves their not conceding at all. Their hazard rate of concession, zero, equals α times the population concession rate (zero, because they are the whole population).

Bayes' Rule. Let z_1 represent 1's reputation (the probability 2 attributes to 1's being a behavioral type) immediately before a concession, in a region where in equilibrium, 1's concession distribution function is smooth. Upon conceding, 1 loses reputation: the posterior is αz_1^0. This formula is a natural extension of Bayes' rule to the continuous setting: in a small time interval $(t; t + \Delta)$, the probability (to first order) that player i concedes is $\lambda_i(t)\Delta$ (where $\lambda_i(t)$ is the conditional probability of concession by player i at t) and the corresponding probability for a behavioral type is $\alpha_i \lambda_i(t)\Delta$.

Although Bayes' Rule does not apply, we deflate player i's reputation by the factor α_i even if she makes a concession that neither her "normal" nor behavioral type is supposed to make at that time, in a particular equilibrium. In our example (see Section 4), this yields a unique bargaining equilibrium. This α rule can be generated exactly by viewing the game as the limit of a sequence of perturbed games in which optimizing players are subject to transitory payoff shocks that occur with extremely low probability, but which lead those players to make concessions of arbitrary sizes. In a *bargaining equilibrium*, defined formally in the Appendix, rational players update beliefs using Bayes' Rule (or, more generally, the α rule discussed in the previous paragraph). After any history they play a best response to those updated beliefs. Behavioral players imitate rational players in every contingency, but with the bias governed by equation 1 above.

Interpretation. If the bargaining game were played only once, without precedent, a behavioral player would have no chance to observe "population behavior" in a given situation, before taking action in that contingency herself. Instead, we assume there is a tradition governing how the game gets played. A behavioral player, rather than optimizing, imperfectly imitates this tradition, introducing the bias α_i described above.

It is natural to ask what happens if behavioral types are more biased against making large concessions than against making smaller concessions. This generalization of the model is explored in section 6.

To sum up, the bargaining game G is defined by the tuple of parameters $\left(Q, \left(x_i^0, z_i^0, \alpha_i, r_i\right)_{i=1}^2\right)$. It is best viewed as an *iterated* war of attrition, in which the number of rounds and the amount conceded at each round are endogenous.

3 Full Concession

We begin by solving an artificial problem in which after some history h, players find themselves R units away from agreement, but each player has only two options at any instant: no concession, or full concession. For example, if four units must be conceded before agreement is reached, we suppress the possibility that either person might make a concession of size one, two or three. We will refer to such games as *full concession games*. Of course, the full concession game coincides with the actual game of interest for $R = 1$, but the solutions for $R \geq 2$ are building blocks in the construction of equilibria of more complex subgames.

Although the details differ, the unique equilibria of full concession subgames look qualitatively just like the unique equilibria of the Abreu-Gul bargaining game.[8] The bargainers play a war of attrition, each gradually becoming more convinced that her opponent is behavioral, because the latter concede more slowly. There comes a time, say T, at which if no concession had yet occurred, player 1 would be sure that 2 is a behavioral type. At that point, if 1 is "normal," she strictly prefers to concede immediately.

It is easy to show (see below) that T must also be the time at which player 2 would be sure player 1 is a behavioral type. That is, in equilibrium, the two posteriors must reach 1 at the same instant. For this to happen, the players' reputations z_i must stand in the right relationship to each other at the beginning of the subgame. We call the set of pairs (z_1,z_2) that satisfy this relationship "the balanced path." If instead 1's reputation z_1 is too low relative to z_2, for example,

[8]The seminal paper by Kreps and Wilson (1982) analyzes both one-sided and two-sided reputation formation. The equilibrium of the latter case is identical to the equilibrium obtained in Abreu-Gul in the subgames following the initial choice of types.

1 must concede to 2 with positive *probability* at the beginning of the subgame. The probability is just high enough so that if (in a particular realization) no concession is made, 1's reputation for being a behavioral type will be enhanced by the amount needed for z_1 and z_2 to stand in the required relationship: z_1 jumps so that the new posteriors lie on the balanced path. Thereafter, until T, each side's concession hazard rate is constant, at a level that makes the opponent indifferent between conceding and waiting. As compared with the (standard) war of attrition with complete information (see, for instance, the seminal work of Maynard Smith (1982), and Hendricks, Weiss and Wilson (1988) for a rigorous development) equilibrium is unique, and furthermore, the normal type of either player concedes with probability 1 by some common finite time T rather than persisting indefinitely. All this is similar to the two-sided reputation formation analysis of Kreps and Wilson (1982).

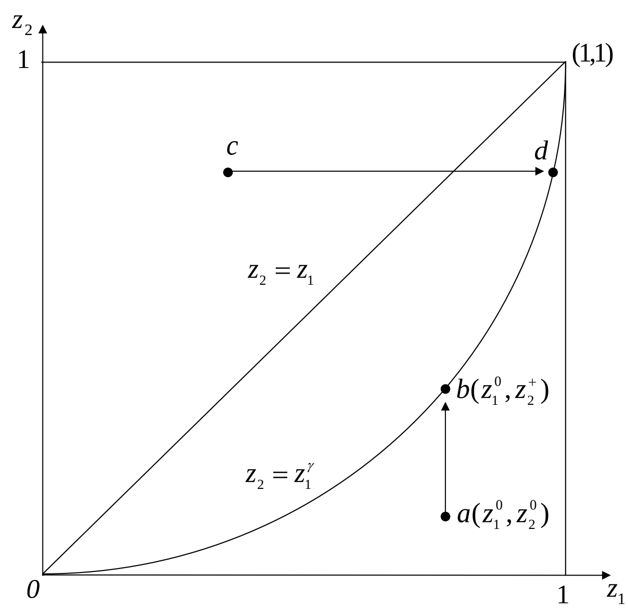

Figure 1: A balanced path and positive probability concessions to reach it.

Let x_i be the current demand of player i. The proof of Proposition 3.1 develops an expression for λ_i, player i's constant hazard rate of concession:

$$\lambda_i = \frac{r_j(Q - x_i)}{x_1 + x_2 - Q}$$

The equation of the "balanced path" alluded to above is shown to be:

$$z_2 = (z_1)^{\frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}}$$

Thus, for example, if $(1 - \alpha_2)\,\lambda_2 > (1 - \alpha_1)\,\lambda_1$ the balanced path is convex and lies below the diagonal. This case is illustrated in Figure 1. If initial reputations $\left(z_1^0, z_2^0\right)$ lie at a point like a below the balanced path, 2 must concede immediately with that probability which increases her reputation to z_2^+ (see point b, directly above a on the balanced path) in the event that no concession is observed. Had initial reputations instead been given by point c, the horizontal jump to d on the balanced path would have been accomplished by 1's conceding with positive probability. If she concedes the game is over. If she is seen not to concede, her reputation is enhanced just enough to leave the reputation pair on the balanced path.

Proposition 3.1 *Consider a full concession game with initial demands x_1 and x_2 and prior beliefs $z_1, z_2 \in (0,1)$. Define $\gamma \equiv \frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}$. There is a unique bargaining equilibrium. In it,*

(i) *if $z_2 = (z_1)^\gamma$ (this is the equation of the balanced path) there are no mass points in the players' strategies; the average hazard rate of concession by player i, λ_i, is constant, and posteriors evolve continuously up the balanced path.*

(ii) *if $z_2 > (z_1)^\gamma$ (we say player 1 is "weak") player 1's concession distribution function has a mass point at time 0, i.e., player 1 concedes at 0 with probability $\omega_1 = 1 - \left(\frac{z_1}{z_2^{1/\gamma}}\right)^{\frac{1}{1-\alpha_1}}$. Conversely, if $z_2 < (z_1)^\gamma$, player 2 concedes with probability $\omega_2 = 1 - \left(\frac{z_2}{z_1^\gamma}\right)^{\frac{1}{1-\alpha_2}}$*

In both cases, conditional upon no concession at time zero the pair of posteriors lies on the balanced path and play proceeds as in (i) above.

Proof. See Appendix.

Some intuition for the form of the balanced path may be useful. In the symmetric case in which interest rates, the α_i factors and the initial demands x_i, are the same for the two players, it is no surprise that the formula reduces to $z_2 = z_1$. Balance is maintained as long as the posteriors are equal.

Consider now a simple departure from symmetry: player 2 has offered 1 more than 1 has offered 2, so 1 has a greater incentive to "cash in now" by conceding. To maintain indifference (and hence, players' willingness to randomize), 2 will need to concede with a higher hazard rate than 1. But then 2's reputation increases faster than 1's if no concessions occur. The only way to have z_1 and z_2 reach 1 at the same time is, therefore, to start z_1 off higher than z_2. The same is true if 2 is more patient than 1; again this requires 2 to concede faster than 1. One sees that either of these asymmetries makes λ_2 greater than λ_1, and the formula

$$z_2 = z_1^{\frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}} \quad \text{where} \quad \lambda_i = \frac{r_j\,(Q - x_i)}{x_j - (Q - x_i)}$$

quantifies the degree of convexity this implies for the balanced path. Figure 2 illustrates the case where everything is symmetric except that 1's demand is 4 whereas 2's demand is 2, and hence $\frac{\lambda_2}{\lambda_1} = 3$.

When play evolves along a balanced path in a full concession game, player i's expected payoff is fixed at what j has already offered (accepting this payoff is one of the alternatives amongst which i is indifferent). But at the beginning of the game, if j is weak, player i's expected utility is higher than j's initial offer, because there is a chance that j will concede to i immediately. This is critical to the understanding of subgames in which partial concessions are possible, the subject of the next section.

4 Partial Concessions

Section 6 will use the results of the last section to give partial characterizations of bargaining equilibria in the model with an arbitrary number of units to be shared. But first we study an example that is tractable enough to allow an explicit account of what happens in each subgame.

In the example there are 5 units to be divided, and each player has initially demanded 4. The players have the same interest rates and their behavioral types are the same: $\alpha_1 = \alpha_2 = \alpha$. If we start them off with the same reputations, that is, $z_1^0 = z_2^0 = z^0$, what happens in equilibrium? If a war of attrition ensues, is it fought one unit at a time? Or do partial concessions weaken one's reputation too much to be profitable?

Any subgame in which 1 and 2 are demanding x and y units, respectively, is called an (x, y) subgame, or sometimes *the* (x, y) subgame (with the understanding that it is parameterized by a pair of initial reputations). By the symmetry of the example, there is no loss of generality in discussing only those (x, y) subgames where $x \geq y$.

Section 3 gives solutions for the (3,3) and (4,2) subgames: in these situations, players are only one step away from agreement, so any concession is a full concession. The other subgames are much more complicated. Here we investigate the

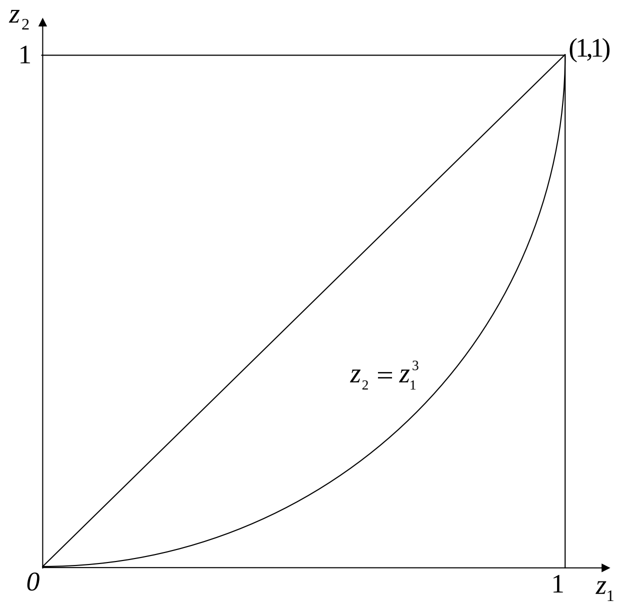

Figure 2: Balanced path in the $(4, 2)$ subgame.

(4,3) subgame. It has some intriguing features, and allows us in Section 5 to solve the (4,4) subgame, the game of ultimate interest.

In the (4,3) subgame, players are two units away from agreement. If player i is going to make a concession, she can either end the game by conceding two units, or concede only one. If player 1 chooses the latter option, they move to the (3,3) subgame, whereas if 2 does so, they move to the (4,2) subgame. How the conceder fares in the new subgame depends upon whether she is weak or strong, which in turn depends on the posterior beliefs at that point.

Our plan of attack is as follows. For any reputation pair z, a concession of either one or two units by player 1 leads to a subgame whose equilibrium payoffs Section 3 has already pinned down. Therefore reputation space can be partitioned (simply, as it turns out) into those points z for which 1 strictly prefers to concede one unit rather than two in the $(4, 3)$ subgame, those from which the opposite is true, and those from which either size of concession is equally attractive. The same can be done for player 2. Now consider, for example, the region in which both players prefer conceding a single unit. It is clear what kind of mixed strategy equilibrium to look for there: a war of attrition in that region must involve each player randomizing between not conceding and conceding one unit. Thus, one first identifies the relevant regions and then analyzes behavior one region at a time. We turn now to the details.

Let z_1 be player 1's reputation immediately prior to a concession. Recall from the discussion in Section 2 that **1's posterior reputation after conceding is scaled down to** αz_1. Since 1 is strong in the (3,3) subgame if and only if the posteriors lie to the right of the balanced path $z_2 = z_1$, 1 will be strong after conceding if in the pre-concession subgame $z_2 < \alpha z_1$. Player 1 strictly prefers conceding one unit rather than two: after conceding one unit, there is a positive probability of receiving an immediate *reciprocal concession* from 2. Had 1 instead conceded two units, she would have precluded the possibility of any gain over what 2 had already offered. (Notice this means, in the terminology introduced in the Introduction, that in the war of attrition before concession, 1's expected payoff is not *residual*: it exceeds what 2 has already offered.) Conversely, if in the pre-concession subgame, $z_2 > \alpha z_1^0$, 1 would be weak after conceding a single unit. In this case, there is no gain to 1 to conceding one unit; if she concedes at all, she concedes both units, ending the game.[9]

These observations are illustrated in Figure 3. If initial reputations are given by point a, for example, a one unit concession by 1 would move her to the (3,3) subgame with reputation pair b, which is to the left of $z_2 = z_1$, so 1 would be weak. Had the initial reputations been given by point c, post-concession reputations in the (3,3) subgame would be the pair d, and 1 would be strong. This would require a probabilistic concession by 2, to end the game or allow the posterior pair to jump up to the balanced path $z_2 = z_1$.

The story for player 2's concession options is similar. From Section 2 we know that the balanced path for the (4,2) subgame is $z_2 = z_1^3$. Following a one unit concession from the (4,3) subgame, 2 will be strong if the posteriors lie above that balanced path. This will occur if and only if her initial reputation z_2 satisfies $z_2 > (1/\alpha) z_1^3$.

Thus, $z_2 = (1/\alpha)\, z_1^3$ is the critical curve dividing reputation space into points from which 2 would prefer to concede a single unit, from points from which 2 will concede fully if at all.

[9]Technically, if 1 were to concede one unit in this circumstance, she would then need to concede again, with positive probability (so that the game would either end, or the posterior pair would jump onto the balanced path for the (3,3) subgame. This would yield 1 the same expected payoff as conceding two units immediately. We ignore this possibility because it has no analog in a discrete-time model, where the delay in making the second concession would be costly.

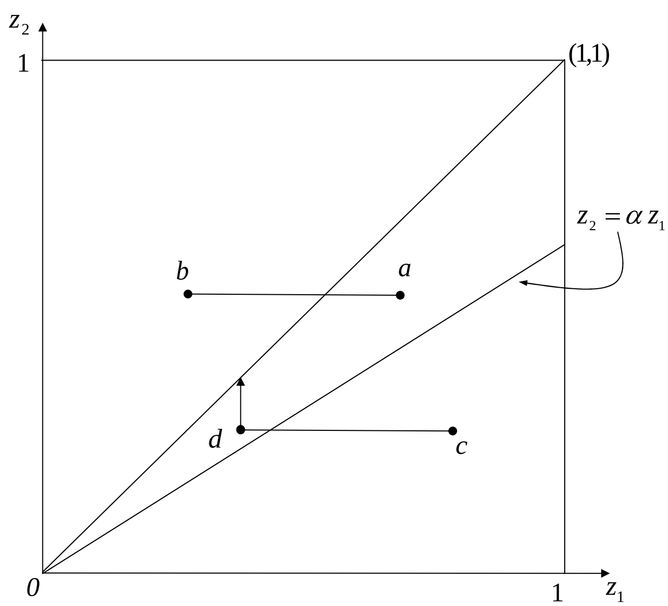

Figure 3: Player 1 concedes two units from point a, but one unit from point c.

Figure 4 plots the critical curves for player 1 and player 2, respectively. Each of the four resulting regions is labelled R_{mn}, where m is the number of units 1 would like to concede, and n is the number 2 would like to concede. *Each region should be viewed as an open set.* On the boundaries between regions, some player will not strictly prefer one concession size to the other.

The most straightforward region is R_{22}. Here, players have no interest in conceding single units, so the full concession analysis of Section 3 applies. The formula

$$z_2 = (z_1)^{\frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}}$$

yields $z_2 = z_1^2$ as the balanced path in this region. From any point in R_{22} below the balanced path (such as e in Figure 5), player 2 must make an immediate probabilistic concession to either end the game or let the reputation pair jump

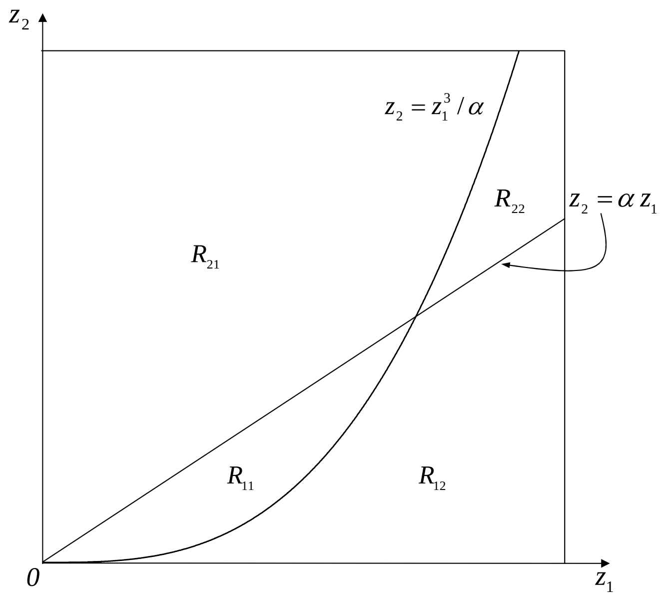

Figure 4: Optimal concession regions in the $(4, 3)$ subgame.

vertically to the path. To the left of $z_2 = z_1^2$ in R_{22}, say, at point f in Figure 5, player 1 must make the probabilistic concession of two units, to restore balance.

In equilibrium a player can concede with positive probability (at an instant) only if she is conceding *fully* at that instant; furthermore, all such lumpy concessions can occur only at the beginning of a subgame.[10]

Application of this principle reveals the solutions for two additional subsets of the parameter space in the (4,3) subgame. Let E be the point of intersection of the

[10]See Lemmas 4.1, 4.2 and 4.3 in the Appendix. Suppose that in equilibrium a player concedes with positive probability at time t. If the concession were partial, she would care about her post-concession reputation. By waiting an instant and conceding just after t, she would enjoy a discrete jump in reputation and be in a stronger position after conceding. Even full concessions can occur only at time $t = 0$. Otherwise, player i's opponent would be unwilling to concede in some interval preceding time t, in which case it is irrational for i to wait until t to concede.

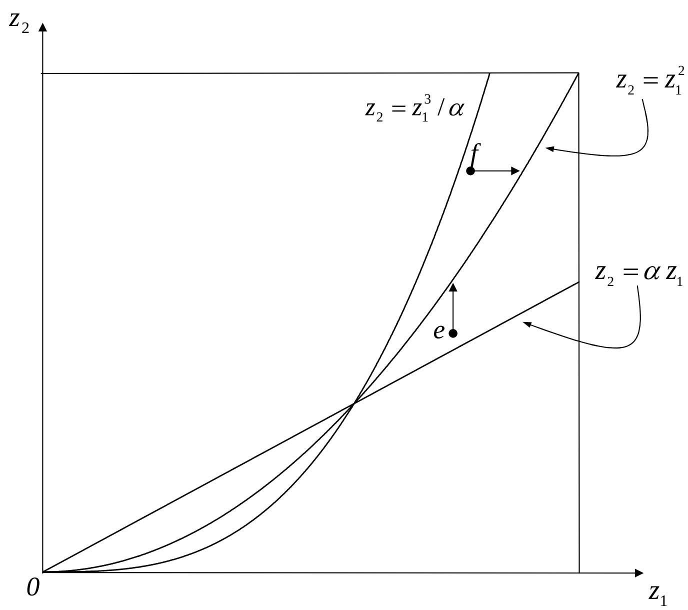

Figure 5: Full concession to reach the balanced path in R_{22}.

two critical lines $z_2 = \alpha z_1$ and $z_2 = (1/\alpha)z_1^3$. Let g be an arbitrary point in R_{12}, with first component greater than E (to the right of the vertical dotted line αE in Figure 6). What happens if players enter the (4,3) subgame with reputation pair g? Either z_2 increases smoothly to 1, or it jumps at some point. The former would require (z_1, z_2) to enter R_{22} somewhere (recall that eventually the reputational pair must reach the point (1,1)), at which point the solution to the region would require an upward jump to the balanced path $z_2 = z_1^2$, a contradiction. So by default, there must be a jump in z_2. Because we are in region R_{12}, z_2 could only jump because of a positive probability of 2's conceding fully at some point, and this point must be the origin of the subgame (by the argument of the preceding paragraph). But the only way for z_2 to evolve smoothly after an initial jump, is for 2 to concede fully at the origin with the right probability so that the posterior pair jumps vertically onto the balanced path $z_2 = z_1^2$ (point h in Figure 6).

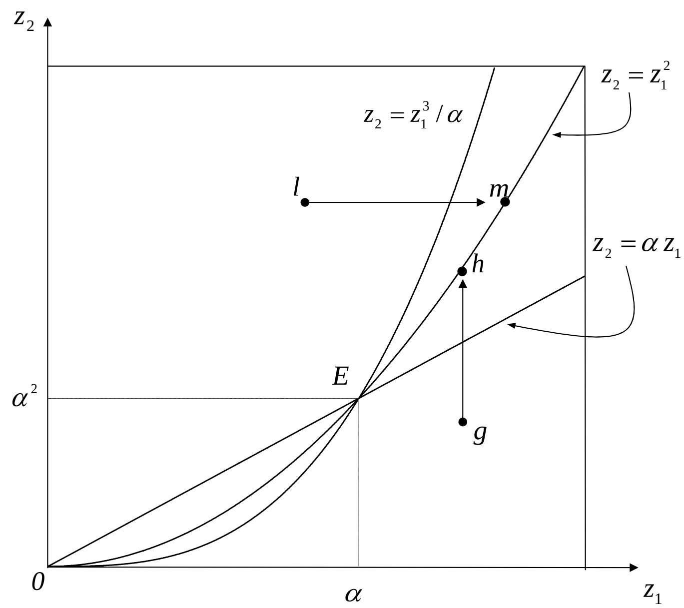

Figure 6: Full concession from outside $O\alpha E\alpha^2$.

The same arguments, with the roles of players reversed, shows that in R_{21} above the horizontal dotted line (Figure 6), 1 initially concedes fully with enough probability so that in the event that the game does not end, the reputation pair jumps rightward onto the balanced path $z_2 = z_1^2$ (from ℓ to m, for example).

What of behavior *within* the box $0\alpha E\alpha^2$? The analysis for this region is both intriguing and intricate; it is carried out in detail in the appendix. Here we simply report the nature of the *unique* equilibrium.

Referring to Figure 7, consider the path from the origin to E labelled $\widetilde{OE}$. It has some of the features of a balanced path: starting at any of its points, reputations evolve smoothly along $\widetilde{OE}$ to E (unless interrupted by a concession of one unit by one party or the other).

But from any point j *above* $\widetilde{OE}$ in the interior of R_{11}, play does *not* jump rightward onto that path. Rather, it proceeds smoothly along a steeper path

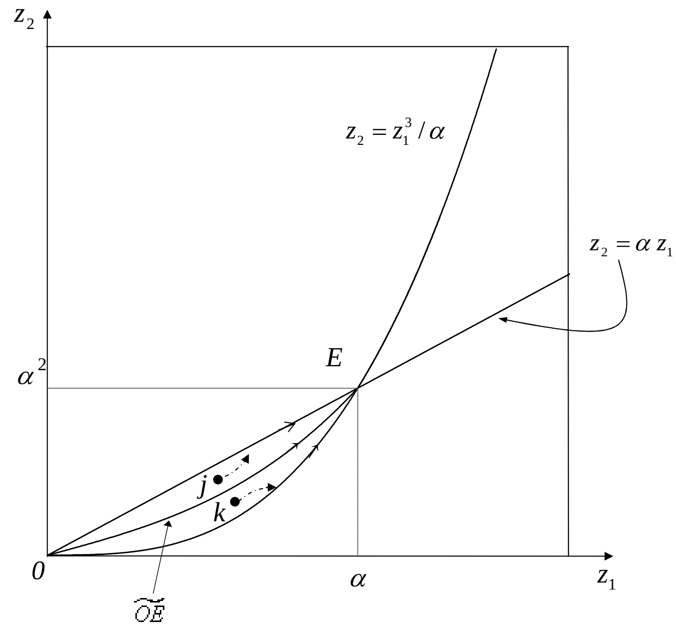

Figure 7: Phase diagram in region R_{11}.

that eventually intersects the upper boundary of R_{11}, unless interrupted by a concession on either side. These steeper paths are phase lines for the dynamics of the evolution of reputations when each side chooses a density of single-unit concessions that makes the other side willing to randomize. Once play hits the upper boundary, or if it starts there, player 1 randomizes at each moment between no concession and a single-unit concession, while 2 randomizes over conceding 0, 1 or 2 units. Notice that this is the first occasion in our analysis in which the amount a player is willing to concede is not determined uniquely.

Similarly, from a point such as k in the interior of R_{11} *below* $\widetilde{OE}$, play evolves via smooth single-unit randomization along a relatively shallow path that intersects the lower boundary of R_{11}. Along that boundary, player 1 may concede one or two units (or more) and player 2 randomizes between not conceding or conceding one unit. It is worth noting that along the boundaries of R_{11}, the weights given to one or two unit concessions are changing, resulting in *non-stationary* wars of attrition that are more complicated than the wars fought elsewhere in the paper.

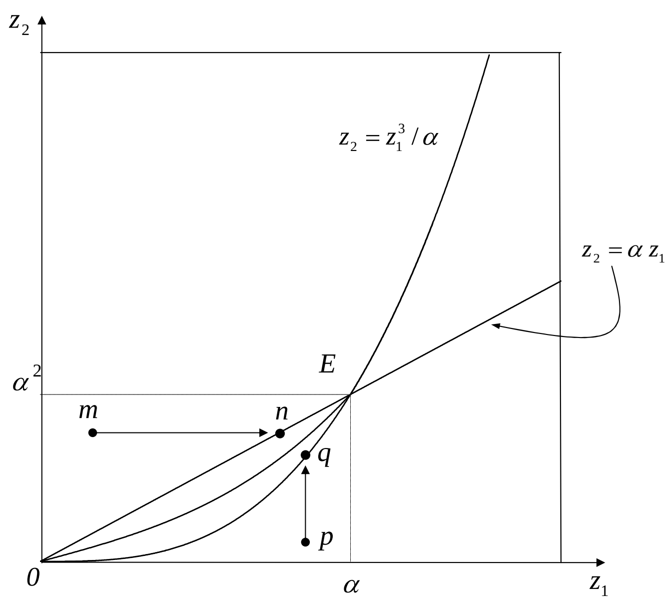

Figure 8:

At a point such as m in figure 8, player 1 makes an instantaneous one-unit concession with sufficient probability that, if no concession is observed, the reputation pair jumps to n on the upper boundary of R_{11}. Analogously, at p in figure 8, player 2 needs to concede probabilistically, either moving play to the $(4, 2)$ subgame or, in the absence of concession, causing the reputation pair to jump vertically to q on the lower boundary of R_{11}.

5 The Equilibrium Path

Having solved for equilibrium behavior in the subgames where players are only one or two steps away from agreement, we can finally address what happens in the full bargaining game. The players begin in the (4,4) subgame, with some initial reputations $z_1^0 = z_2^0$. From this position, each player has four concession options: three units, two units, one unit, or none at all. Fortunately, at each value of z, the analysis of Section 4 lets us eliminate two of these possibilities.

We show in the appendix that equilibrium in the (4,4) subgame is unique. Starting from any initial reputation pair on the diagonal, the unique equilibrium is symmetric and entails continuous concession by both players at identical rates. Thus in the absence of concession, posterior reputations evolve smoothly up the diagonal toward the point (1,1). We focus here on behavior along the equilibrium path. The discussion below characterizes the optimal amount of concession at different points on the diagonal, building upon the discussion of the (4,3) and (4,2) subgames presented earlier.

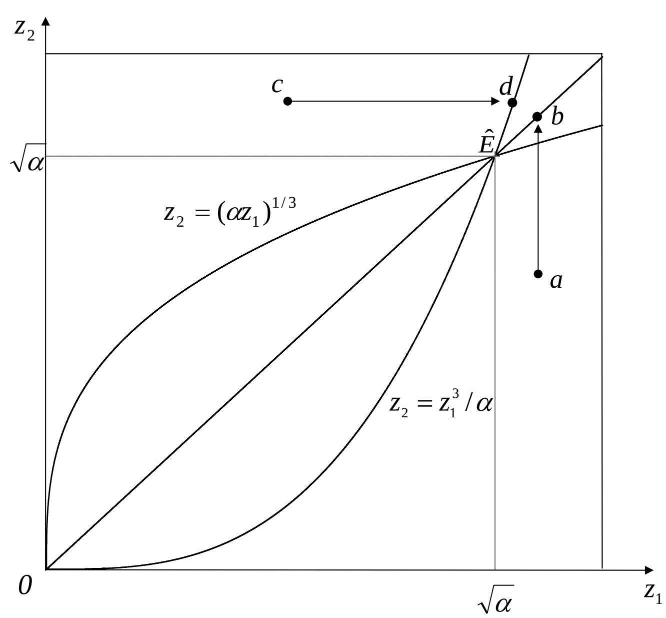

Figure 9: Full concession outside $O\sqrt{\alpha}\widehat{E}\sqrt{\alpha}$.

Starting from points on the diagonal (see Figure 9) above $(\sqrt{\alpha}, \sqrt{\alpha})$, a player

(2, for specificity) does not wish to concede one unit. If she did, her post-concession reputation would lie on the $z_2 = \alpha z_1$ line, which is below the balanced path for the (4,3) subgame, namely $z_2 = z_1^2$. Being weak, 2 would have to (probabilistically) concede two more units right away. Again appealing to the discrete-time analog of our game (recall footnote 10), we require that 2 should have ended the game by conceding three units in the first place.

Similarly, above $(\sqrt{\alpha}, \sqrt{\alpha})$ on the diagonal, 2 does not want to concede two units. Such a concession would put her in the (4,2) subgame on the $z_2 = \alpha z_1$ line, which, to the right of $z_1 = \sqrt{\alpha}$, lies below the balanced path $z_2 = z_1^3$ for that subgame. As before, 2 would be in a weak position, and should have just ended the game with a three-unit concession.

Above $(\sqrt{\alpha}, \sqrt{\alpha})$, then, one is in a "full concession" subgame, and the formulae of Section 3 apply. The players engage in a war of attrition, each randomizing between waiting and conceding fully. The overall hazard rate for concession by either player in this region is given by the formula

$$\lambda_j(t) = \frac{r(Q - x_i)}{x_i - (Q - x_j)}$$

that is,

$$\lambda_1(t) = \frac{r}{3} = \lambda_2(t)$$

Below $(\sqrt{\alpha}, \sqrt{\alpha})$ on the diagonal, it turns out that **conceding two units is always better than conceding one**. First, look at points to the right of $z_1 = \alpha$, but to the left of $z_1 = \sqrt{\alpha}$. Here, if 2 concedes one unit, she goes to the (4,3) subgame with a reputation that leaves her weak (because $z_2 = \alpha z_1$ lies below the balanced path $z_2 = z_1^3$). If instead she had conceded two units, she would have arrived in the (4,2) subgame in a strong position: $z_2 = \alpha z_1$ lies above $z_2 = z_1^3$, the balanced path for the subgame. Thus, conceding two units is superior to making a minimal concession of a single unit or a maximal concession of three (the latter would end the game without a chance of a reciprocal concession).

Finally, what if z_1 is less than α? If 2 concedes one unit from the (4,4) subgame, in this region play proceeds upward along the $z_2 = \alpha z_1$ line, by means of simultaneous randomization by the players. Along that path, one of 2's optimal strategies is to concede a single unit. Thus, for the purposes of computing her expected utility in this circumstance, we may suppose she concedes that additional unit immediately. This reduces her reputation further by a factor of α. In effect, 2 has ended up in the (4,2) subgame, with a reputation $z_2 = \alpha^2 z_2^0$. But she could have reached the same subgame with higher reputation $z_2 = \alpha z_2^0$, by simply conceding two units (from (4,4)) in the first place. The analysis of Section 3 tells us this would afford her a higher expected utility.

In any region where both players' optimal concession is two units,

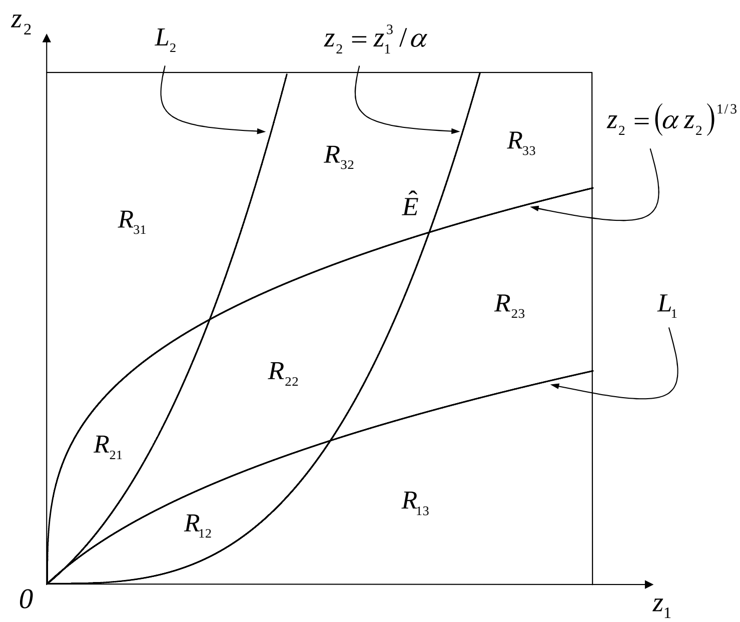

Figure 10: Optimal concession regions in the $(4,4)$ subgame.

$$u_2(t) = \int 3e^{-rt} dF_1(t) + e^{-rt}(1 - F_1(t))\left[1 + \omega_1(z_1(t), \alpha z_2(t))\right]$$

where $\omega_1(x_1, x_2) = 1 - \left[\frac{x_1}{(\alpha x_2)^{1/3}}\right]^{1/1-\alpha}$. Define $u_1(t)$ symmetrically.

The requirement $u_i'(t) = 0$ yields

$$(1 + \omega_1) r = \lambda_1 + \frac{\lambda_2}{3}(1 - \omega_1) \quad \text{and} \quad (1 + \omega_2) r = \lambda_2 + \frac{\lambda_1}{3}(1 - \omega_2)$$

$$\Rightarrow \lambda_1(t) = \lambda_2(t) = \frac{3(1 + \omega_1) r}{(4 - \omega_1)}.$$

The above gives the equilibrium rates of concession for the two players at points on the diagonal (z,z) smaller than $(\sqrt{\alpha}, \sqrt{\alpha})$.

To summarize, let us consider the possible paths that could be followed by two normal types facing one another, starting with symmetric reputations. For a time interval of random duration, nothing happens in the bargaining game, but their reputations move continuously up the 45° line. At some point one player, say 2, will make a concession. If this happens when reputations are greater than $\sqrt{\alpha}$, the concession will be complete, and the game ends. This eventuality is represented by the open circle o on the diagonal in Figure 9. But if the concession occurs before $\sqrt{\alpha}$ is reached, 2's concession will be two units, and 1 will respond immediately by making a probabilistic concession of one unit, which either ends the game, or (if no concession is made) puts them on $z_2 = z_1^3$, the balanced path in the $(4, 2)$ subgame.

The above discussion has focussed on behavior along the diagonal. We now turn to initial reputational pairs which lie off the diagonal. As in the detailed analysis of the (4,3) subgame we need to identify regions $\hat{R}_{mn},\ m, n = 1, 2, 3$, in the z_1, z_2 plane in which players 1 and 2 optimally concede m and n units, respectively. These regions are diagrammed in Figure 10. Details of the derivation of these regions are provided in the Appendix. As in the preceding section, each of these regions should be viewed as an open set.

Equilibrium behavior is summarized in Figures 11 & 12. Normally it is crucial in a war of attrition for each party to randomize in such a way as to make the other party indifferent between giving in and holding out. With *partial concessions* and reputational concerns, however, it is conceivable that player i randomizes between conceding and making a partial concession, while over the same interval of time, player j simply waits rather than randomizing. What does i gain by waiting, that offsets her impatience and makes her willing to randomize? As time passes without a concession, i's reputation improves (while j's remains constant). Thus she is stronger in the post-concession subgame when she delays. We refer to this phenomenon of one player conceding randomly over on interval while the other waits as *solo concession*.

If player 1 is making a solo concession, the rate at which she concedes is chosen so that her reputation increases just quickly enough to keep her indifferent about conceding. This might be fast enough to make player 2 strictly prefer to wait, or instead be too slow for 2 to be willing to wait for a concession. In the $(4, 4)$ subgame, the locus of points at which 2 is exactly indifferent about waiting for 1's solo concession is labeled L_2^* in Figures 11 & 12. To the left of L_2^*, player 2 won't wait (so in that region, solo concession by 1 cannot occur in equilibrium).

From a point like c in Figure 11 player 1 concedes lumpily (and by three units, i.e. fully) so that conditional upon not conceding, her posterior reputation jumps to d. Thereafter play evolves along the curve $(\alpha z_2)^{1/3}$ until point e. Both players concede continuously along this path, player 2 conceding one unit and player 1 *randomizing* between conceding two and three units at a rate which is calibrated so that reputations evolve along the boundary between regions $\hat{R}_{31}$ and $\hat{R}_{21}$. At

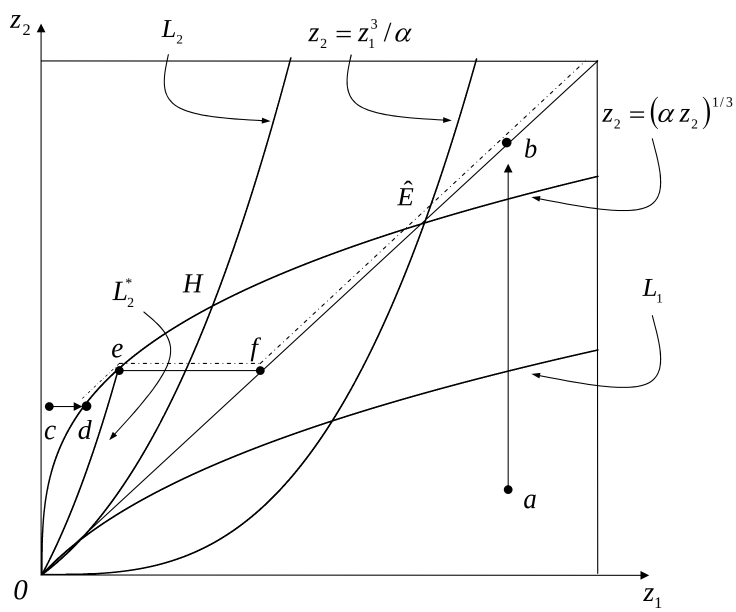

Figure 11: Equilibrium in the $(4,4)$ subgame.

point e, solo concession by player 1 resumes until point f, after which both players concede continuously, and reputations evolve along the diagonal.

From a point like a, below the diagonal and outside the box $0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$, player 2 concedes lumpily with probability such that conditional upon not conceding, her posterior reputation jumps to b. Thereafter both players concede continuously and reputations evolve along the diagonal.

Now refer to figure 12. From a point like i to the right of L_2^* and left of the diagonal, solo concession by player 1 continues until point j on the diagonal, after which play evolves along the diagonal. Finally from a point like k to the right of $(\alpha z_2)^{1/3}$ and to the left of L_2^*, both players concede continuously (player 1 by *two* units and player 2 by *one* unit) and the path of posterior reputations increases monotonically until either (depending on the precise location of k) the L_2^* boundary or the $(\alpha z_2)^{1/3}$ boundary is reached. In the former case we have

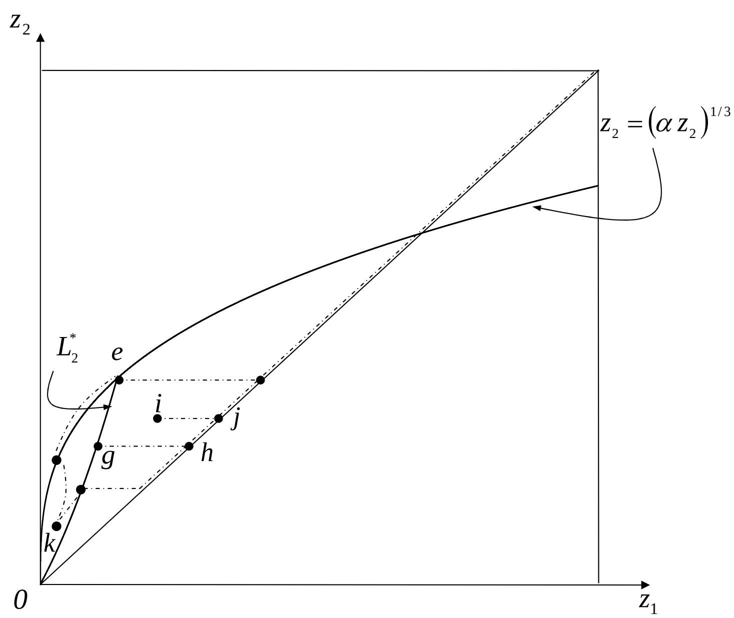

Figure 12: Equilibrium in the (4, 4) subgame.

solo concession by 1 until the diagonal is reached, and in the latter, evolution of reputation along $(\alpha z_2)^{1/3}$ to point e, followed by solo continuous concession by 1 until the diagonal is reached.

Details of these derivations are in the Appendix.

Limit Results and Comparative Statics

Although our interest lies primarily in uncovering patterns of behavior such as reciprocity and concessions of intermediate size, a few comparative statics exercises shed some further light on the model. The propositions below focus on the welfare consequences of changes in reputations and degrees of behavioral bias, and explore the continuity of the model as z approaches 0 and, respectively, as α approaches 0 or 1. Proofs are provided in the Appendix.

The first set of results fix parameters $r_1 = r_2$ and $\alpha_1 = \alpha_2$ and study the dependence of player i's expected payoff $v_i(z_1, z_2)$ on the initial reputations z_1 and z_2. Limit results as reputations approach zero are of particular interest because they concern payoffs of slight perturbations of a standard bargaining model without behavioral biases.

If the bargaining equilibrium were efficient, players' expected payoffs would sum to 5, the total units available. Even as the initial probability of behavioral types approaches 0, along the main diagonal, Proposition 5.1 says that $v_1(z,z)+v_2(z,z)$ is only 4. We expect that the discontinuities implicit in propositions 5.1 and 5.7 are associated with the discrete nature of this bargaining problem. Our conjecture is that one would have continuity in a model without indivisibilities.

Proposition 5.1 $\lim_{z\to 0} v_i(z,z) = 2, \quad i = 1,2$

Interestingly, in the neighborhood of $(z_1, z_2) = 0$, bargainers with equally matched reputations result in the most aggregate inefficiency. When z_1 and z_2 approach zero along a ray with $\frac{z_1}{z_2} = a$ for some positive number a, player 1's limiting payoff can range from 2 to 3 depending on the value of a.

Proposition 5.2 *Let* $(z_1^n, z_2^n)_{n=1}^{\infty}$ *be a sequence of reputational pairs. Suppose*

$$\lim_{n\to\infty} z_1^n = \lim_{n\to\infty} z_2^n = 0 \text{ and } \lim_{n\to\infty} \frac{z_2^n}{z_1^n} = a \in (0,\infty).$$

Then if $a < 1$,

$$\lim_{n\to\infty} v_1(z_1^n, z_2^n) \geq \left(3 - (a)^{1/1-\alpha}\right) \in (2,3) \text{ and } \lim_{n\to\infty} v_2(z_1^n, z_2^n) = 2.$$

Conversely, if $a > 1$,

$$\lim_{n\to\infty} v_1(z_1^n, z_2^n) = 2 \text{ and } \lim_{n\to\infty} v_2(z_1^n, z_2^n) \geq \left(3 - \left(\frac{1}{a}\right)^{1/1-\alpha}\right).$$

By way of contrast to Proposition 5.2, one gets continuity in the limit as (z_1, z_2) approaches $(1,1)$. Payoffs become residual, and the expected aggregate inefficiency is 3.

Proposition 5.3 *Let* $(z_1^n, z_2^n)_{n=1}^{\infty}$ *be a sequence of reputational pairs. Suppose* $\lim z_i^n = 1$, $i = 1,2$. *Then* $\lim_{n\to\infty} v_i(z_1^n, z_2^n) = 1$. *Both players' payoffs are residual in the limit.*

Intuition suggests that each player would like to have a high reputation (for being behavioral) and that this advantage comes at the expense of her rival. This is indeed true, with one interesting exception.

Let $\hat{\hat{R}}_{21}$ represent the region of $\hat{R}_{21}$ which lies below L_2^* and let $\hat{\hat{R}}_{12}$ represent the corresponding region below the diagonal.

Proposition 5.4 *For all* $(z_1, z_2) \notin \hat{\hat{R}}_{21} \cup \hat{\hat{R}}_{21}$,

$$\frac{\partial v_i(z_1, z_2)}{\partial z_i} \geq 0 \quad i = 1, 2$$

$$\frac{\partial v_i(z_1, z_2)}{\partial z_j} \leq 0 \quad i \neq j, \quad i = 1, 2$$

Moreover, these inequalities are strict whenever $v_i(z_1, z_2) > 1$*, that is, when player i's payoffs are non-residual. For* $(z_1, z_2) \in \hat{\hat{R}}_{21}$*,* $\frac{\partial v_2(z_1,z_2)}{\partial z_2} > 0$ *as above, but* $\frac{\partial v_2(z_1,z_2)}{\partial z_1} > 0$*. Conversely for* $(z_1, z_2) \in \hat{\hat{R}}_{12}$*.*

Propositions 5.5 and 5.6 fix $z_1 = z_2 = z$ and $\alpha_1 = \alpha_2 = \alpha$, and investigate $\hat{v}_i(r_1, r_2)$, player i's expected utility as a function of the two interest rates. When $r_1 = r_2$, the common value of the interest rate does not matter at all for players' welfare. Students of wars of attrition will not be surprised; when players are more patient, concessions simply occur more slowly, so that waiting long enough to receive a concession with a given probability is just as costly as it would have been in a world with very impatient players.

Proposition 5.5 *The equilibrium payoff function* $\hat{v}(r, r)$ *is independent of* $r > 0$*:*

$$\hat{v}(r, r) = \begin{cases} 2 - (\frac{z}{(\alpha z)^{1/3}})^{1/(1-\alpha)} & for \quad z < \sqrt{\alpha} \\ 1 & for \quad z \geq \sqrt{\alpha} \end{cases}$$

Finally fix $z_1 = z_2$, $r_1 = r_2$ and consider expected payoff functions $\tilde{v}_i(\alpha_1, \alpha_2)$. Let $(\alpha^n)_{n=1}^{\infty}$ be a sequence of behavioral bias parameters. If the biases become arbitrarily severe, the model asymptotically approaches the Abreu-Gul model with exogenous types who inflexibly demand 4 units each.

Proposition 5.6 *For* $i = 1, 2$ *if* $\alpha^n \to 0$*, then* $\lim_{n\to\infty} \tilde{v}_i(\alpha^n, \alpha^n) = 1$*.*

On the other hand, even with arbitrarily mild biases, bargaining equilibrium is bounded away from efficiency: in the limit the sum of the expected utilities is 4, whereas 5 units were to be shared.

Proposition 5.7 *For* $i = 1, 2$ *if* $\alpha^n \to 1$*, then* $\tilde{v}_i(\alpha^n, \alpha^n) = 2$*.*

6 Extensions

A vivid picture of an iterated war of attrition emerges from the bargaining equilibrium studied in the preceding two sections. Where possible, a player chooses a partial concession that has a good chance of inducing a reciprocal concession. One is prompted to ask how much the observed patterns depend on special features of the class of example we chose. In particular, in more general settings will equilibria exhibit reciprocity, non-residual payoffs and concessions of intermediate size?

This section offers some tentative answers to these questions. The model considered here is exactly like that of Sections 4 and 5 except in two important respects. First, as in Section 3, the number of units to be divided, Q, can be any positive integer, and initial demands x_1^0 and x_2^0 are arbitrary integers less than Q, whose sum exceeds Q. Secondly, while there is still only one behavioral type for each player, the description of the type allows some sizes of concession to be avoided more than others (relative to the population average). For example, it is natural to regard a large concession as a more convincing signal of "normalcy" than a small concession would have been.

Formally i's behavioral type is now described by a function $\alpha_i : \{1, ..., Q\} \to (0, 1)$, where $\alpha_i(k)$ is the factor by which the behavioral type scales down the probability of conceding k units after a given history, compared to the population average. For the same reasons as in Section 2, this means that in a war of attrition situation, if player i with behavioral type α_i makes a k-unit concession, her reputation falls from z_i to $\alpha_i(k) z_i$. The bargaining game is defined and denoted exactly as in Section 2, except that player i's type α_i is now a *function*. The definition of bargaining equilibrium also extends in an obvious way.

Two non-exhaustive classes of behavioral types are helpful in understanding what equilibria look like.

Definition 6.1. The behavioral type α_i is *log superadditive* if for all $k, \ell \in \{1, ..., Q\}$, $\alpha_i(k + \ell) > \alpha_i(k)\alpha_i(\ell)$. When the strict inequality above is reversed, α_i is said to be *log subadditive.*

Recall that in earlier sections $\alpha_i(k)$ was constant, a strong case of log superadditivity. Suppose player i of type α_i wants to concede two units. In the log superadditive case, she is left with a better reputation if she makes one two-unit concession than she would were she to make two consecutive concessions each of size 1. The opposite is true in the log subadditive case. This suggests that log subadditivity favors small concessions over large, and log superadditivity does the opposite. This intuition is appropriate, but needs some qualification. If $\alpha_i(1)$ is sufficiently small, for example, even in the log subadditive case small concessions are never made: 1 randomizes between not conceding and conceding *fully.*

Proposition 6.1 gives a rough sufficient condition for player i's payoff to be *nonresidual*, that is, to exceed strictly the amount left over when her opponent's demand is subtracted from Q. A longer argument can establish a tighter sufficient condition for the log subadditive case.

Proposition 6.1 *Fix an equilibrium of the bargaining game and consider any (non-terminal) history h with players' outstanding demands and reputations given* *by the vector* $(x_1, x_2; z_1, z_2)$*. Then player i's payoffs are non-residual after h if*

$$z_i > (z_j)^{\frac{\left(1-\alpha_i^k\right)r_j\left(x_j-1\right)}{\left(1-\alpha_j^1\right)r_i\left(Q-x_j\right)}} \quad \text{where} \quad k = (x_1 + x_2 - Q - 1)\,.$$

This condition will not be hard to satisfy, especially in the early stages of the game. If Q is large (the integer unit of measurement is small) and if initial demands are close to Q, then $\frac{x_j-1}{Q-x_j}$ is large. Absent big differences in the ratio of initial reputations or rates of impatience for the two players, and if $\alpha_i(k)$ is not much smaller than $\alpha_j(1)$ (conversely $\alpha_j(k)$ not much smaller than $\alpha_i(1)$) then the above inequality and the corresponding one for player j will be easily satisfied. The proof of the result is immediate: the inequality is simply the condition for a player to be strictly strong after conceding almost fully (i.e. fully less one unit).

To say something about reciprocity, we need to restrict attention to what we call *regular Markov equilibria*, defined below. Since we have shown nothing about existence of regular Markov equilibria[11] the proposition that follows might be (we don't think so!) vacuous.

Definition 6.2. A *Markov bargaining equilibrium* is a bargaining equilibrium which in addition satisfies the requirement that players' behavior after any history h depends only on the state after that history, where a state is comprised of the players' outstanding demands $x(h) = (x_1(h), x_2(h))$ and reputations $z(h) = (z_1(h), z_2(h))$ after that history.

One expects that in most circumstances, a stronger reputation will increase a bargainer's expected payoff, ceteris paribus. But it need not do so *strictly*: even in our example of Section 4, we saw that one step away from agreement, it makes no difference to 1 whether she is weak or very weak: either way, her expected payoff is just what 2 has already offered her.

Definition 6.3. (Regularity of Markov bargaining equilibrium) Fix a Markov bargaining equilibrium σ. Then players' expected payoffs after any history depend only on the state (x, z) after that history. Denote player i's payoff $v_i(x, z)$. σ is a *regular* Markov bargaining equilibrium if v_i is strictly increasing in z_i when player i's payoffs are non-residual (that is, when $v_i(x, z) > Q - x_j$) at (x, z).

Proposition 6.2. *(Reciprocity) Suppose* σ *is a regular Markov bargaining equilibrium and suppose that player i's payoff is non-residual at some state* (x, z)*. Then if player i's type* α_i *is log superadditive, an equilibrium concession by player i (at state* (x, z)*) must be followed by a reciprocal response by player j. Conversely if player i's type is log subadditive, player i's equilibrium concession is the minimal size possible (one unit) and does not necessarily result in a reciprocal response by player j.*

[11]See Maskin and Tirole (1997) for the problems in establishing existence of Markov perfect equilibrium even in relatively simple finite games.

7 Conclusion

This paper is concerned with bargaining theory, on the one hand, and the novel device of endogenous perturbations, on the other. Introducing endogenous types with behavioral biases stated in simple, relative terms yields a complex pattern of bargaining dynamics. The resulting equilibrium is an *iterated war of attrition.* Along the equilibrium path, one frequently observes an initial war of attrition ending in a *partial* concession by player 1, say, followed by another war of attrition. In the initial war, player 1 strictly prefers not to concede fully. His reward for conceding partially is the *chance* that he may receive an immediate *reciprocal concession* from player 2. This is consistent with what happens in many bilateral bargaining situations: after a long stalemate, a breakthrough in negotiations occurs in which both sides soften their stances noticeably in a short period of time.

In the model, as in the real world, bargaining need be neither a long, slow slide towards mutual agreement, nor an "all or none" event taking place in one step. While some concessions, especially very small ones, would weaken a player's bargaining position, a carefully measured change in demand may be rewarded. A rough theory of optimal concession size emerges. In the family of examples we solve fully, players favor larger concessions when they are at a noticeable reputational disadvantage relative to their rivals, or when both reputations are high.

We hope that endogenous perturbations will prove a parsimonious way of capturing behavioral biases in static models. Perhaps their most powerful applications will be to dynamic models where, as in the bargaining model here, their introduction provides reputational reasons for rational players to adopt intricate strategies. Beyond bargaining models, we think that allowing for perceptual biases in repeated games could be a tractable way of exploring the inefficiencies that naturally accompany ambiguities in how the gains from cooperation should be shared.

8 Appendix

The appendix provides some formal definitions and details of arguments left incomplete or omitted from the text.

Section 2.

Histories

An *M-episode history* is a finite sequence of pairs $(t^n, x^n)_{n=1}^{M}$, where the pair $(t^n, x^n) \in (I\!R_+ \cup \infty) \times \{1, 2, ..., Q\}^2$ represents a concession "episode" at time t^n, and the sequence satisfies the requirements:

(i) $t^n \geq t^{n-1} \quad n = 1, 2, ..., M$

(ii) $x_i^n \leq x_i^{n-1} \quad i = 1, 2$ with strict inequality for at least one i

(iii) $x_1^n + x_2^n > Q \qquad n = 1, 2, ..., M-1$

Condition (i) formalizes the idea that the sequence records concessions in their order of occurrence, but the weak inequality permits one concession at time t to be followed instantaneously by another concession at time t. Condition (ii) ensures that at least one player actually makes a concession and Condition (iii) says that concessions cease when compatibility of demands has been achieved.

We will refer to an M-episode history for which $x_1^M + x_2^M \leq Q$ or $t_M = \infty$ as a *terminal* history and as a *non-terminal* history otherwise. In the event that the game ends with *simultaneous* concessions $x_i^M < x_i^{M-1}$, $i = 1, 2$, by the two players the final division is taken to be $\left(x_1^M, Q - x_1^M\right)$ or $\left(Q - x_2^M, x_2^M\right)$ with equal probability: the unclaimed surplus is awarded randomly.

Bargaining Equilibrium

Let h be a T-period history in which a concession has just occurred at T. Let $z_i(h)$ denote the posterior probability that player i is behavioral, immediately following the concession at T. Let $(F_i(. \mid h), \kappa_i(. \mid h))$ govern player i's concession behavior until the next concession episode. That is, $F_i(t \mid h)$ is the probability that player i will make a new concession in the interval $[t, t+T]$, conditional upon j not conceding prior to $t+T$, and $\kappa_i(t \mid h)$ is a probability distribution over possible reduced demands (i.e. concessions), conditional upon player i conceding at $t+T$. The discussion in Section 2 implies that if a concession occurs at $t+T$ (and if $F_i(. \mid h)$ does not have a mass point at t) then the posterior probability (call it $\widehat{z}_i$) that player i is behavioral after observing concession at $T+t$, is

$$\widehat{z}_i = \frac{\alpha_i z_i(h)}{(1 - F_i(t \mid h))^{1-\alpha_i}} \tag{2}$$

where

$$\frac{(1 - H_i(t \mid h))}{(1 - F_i(t \mid h))} z_i(h) \;=\; \frac{(1 - F_i(t \mid h))^{\alpha_i}}{(1 - F_i(t \mid h))} z_i(h) \tag{3}$$

is the posterior probability that player i is behavioral just prior to the concession at $T + t$. The same definitions apply when h is the null history ($h = \emptyset$) in which case $z_i(h = \emptyset) = z_i^0$ is the prior probability, at the start of play, that player i is behavioral.

Note that the function $G_i(. \mid h)$ governing a normal player i's concession behavior is pinned down by:

$$(1 - F_i) = z_i(1 - H_i) + (1 - z_i)(1 - G_i) = z_i(1 - F_i)^{\alpha_i} + (1 - z_i)(1 - G_i)$$

where for transparency we have suppressed the dependence of z_i, F_i, etc., on h.

A strategy for player i is a collection of functions $(F_i(. \mid h), \kappa_i(. \mid h))_h$ where h ranges over all possible non-terminal histories of concession episodes. Since, by assumption, concessions once made cannot be reversed, any particular history cannot have more than a finite number of concession episodes (precisely, at most, $(x_1^0 + x_2^0 - Q - 1)$ episodes). The posterior probabilities that the players are behavioral following a concession are inductively determined via the formulae (2) and (3) depending upon which applies at the inductive step in question.

Let $x_i(h)$ denote player i's current demand after history h. Let $\psi_i(t, x_i \mid h)$ denote the payoff to a normal player i of conceding at time t to the demand $x_i < x_i(h)$ conditional upon the history h, and given that subsequent behavior is governed by $(F_l, \kappa_l)_{l=1,2}$. Let $\mu_i(h)$ denote the product measure defined by $(G_i(. \mid h), \kappa_i(. \mid h))$ and

$$A_i(h) = \{(t, x_i) \mid \psi_i(t, x_i \mid h) = \sup_{(t', x')} \psi_i(t', x_i' \mid h)\}$$

The pair $(F_l, \kappa_l)_{l=1,2}$ define a bargaining equilibrum if for $i = 1, 2$ and all non-terminal histories h,support $\mu_i(h) \subseteq A_i(h)$.

<u>**Section 3**</u>

Proof of Proposition 3.1

As noted in the text the *full* concession case turns out to be very similar analytically (despite a quite different definition of type) to the model analyzed in Abreu-Gul (2000) - hereafter (A-G). The proof below is adapted from the proof of Proposition 1 in A-G. A number of properties of equilibrium distribution functions (continuity, strict monotonicity, differentiability), are in turn familiar from previous analyses and applications of wars of attrition (e.g. Hendricks, Weiss and Wilson (1988)).

Suppose that σ is a bargaining equilibrium of the full concession game. Let F_i be the *average* (equilibrium) distribution function over concession times for player

i; let G_i and H_i be the corresponding functions for normal and behavioral types respectively. Let $T_i = \inf \{t \mid G_i(t) = 1\}$. (If $G_i(t) < 1$ for all $t > 0$, set $T_i = \infty$.)

Step 1. $T_i > 0$ *for* $i = 1, 2$.

Proof. Suppose $T_i = 0$. Then, a profitable deviation is for player i to wait a moment, instantly convince j that she is behavioral for sure, and in consequence be conceded to immediately by a normal player j. ■

Steps 2 and 6 are essentially identical to Steps (a)-(e) of the proof of Proposition 1 of AG. Their proofs are omitted.

Step 2. $T_1 = T_2 \equiv T$.

Step 3. F_i *is continuous at all* $t \in (0, T]$

Step 4. $\forall t'' > t' \geq 0$, *if* $F_i(t'') > F_i(t')$ *then* $F_j(t'') > F_j(t')$

Step 5. *For all* t', t'' *such that* $T > t'' > t' \geq 0$, $F_i(t'') > F_i(t')$.

Step 6. $(0, T] \subseteq A_i$

Let $u_i(t)$ denote the expected payoff to player i from conceding at time t. Then

$$u_i(t) = \int_0^t e^{-r_i s} x_i dF_j(s) + e^{-r_i t}(1 - F_j(t))(Q - x_j) \tag{4}$$

Let $A_i = \{t \mid u_i(t) = \max_{s \geq 0} u_i(s)\}$

Step 7. It follows that u_i is constant, hence differentiable, on $(0, T]$,and therefore F_j is differentiable on this range also. Differentiating u_i,

$$u_i'(t) = 0 = e^{-rt} x_i f_j(t) - r e^{-rt}(1 - F_j(t))(Q - x_j) - e^{-rt} f_j(t)(Q - x_j) = 0$$

$$\Rightarrow \frac{f_j(t)}{1 - F_j(t)} = \frac{r(Q - x_j)}{x_i - (Q - x_j)} \equiv \lambda_j, \quad \text{a constant.}$$

The distribution function F_j has a *constant* hazard rate λ_j. This is intuitive, and indeed very familiar from classical analyses of the war of attrition. Integration yields $F_j(t) = 1 - c_j e^{-\lambda_j t}$, where $c_j \in (0, 1]$ is a constant of integration uniquely determined by equilibrium conditions. Player j has no mass point at $t = 0$ if and only if $c_j = 1$.

By definition

$$\left(1 - z_j^0\right)(1 - G_j(t)) + z_j^0 (1 - H_j(t)) = 1 - F_j(t)$$

where z_j^0 is the prior probability that player j is behavioral at time zero. By equation (1) which we rewrite below

$$1 - H_j(t) = (1 - F_j(t))^{\alpha_j}$$

To find the T_j at which the normal type of player j will just finish conceding with probability 1, we solve

$$\left(1 - z_j^0\right) \cdot 0 + z_j^0 \left(1 - F_j\left(T_j\right)\right)^{\alpha_j} = 1 - F_j\left(T_j\right)$$

When player j has no mass point at $t = 0$ (i.e. $c_j = 1$), this directly yields

$$T_j = \frac{-\log z_j}{\left(1 - \alpha_j\right)\lambda_j}$$

As argued above, in equilibrium $T_1 = T_2$.

Hence, if there were no mass points at $t = 0$ in either player's concession function, this would imply that the priors satisfy:

$$z_2 = (z_1)^{\frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}} \equiv z_1^{\gamma}, \quad \text{where} \quad \gamma = \frac{(1-\alpha_2)\lambda_2}{(1-\alpha_1)\lambda_1}$$

This is the equation of the *balanced path.* For $t > 0$, equilibrium requires that the posteriors (z_1, z_2) lie on the path. If the priors at $t = 0$ do *not* lie on the path, the normal type of either 1 or 2 must have a mass point in her concession distribution at $t = 0$, so that the game either ends with a concession at 0, or the posterior pair jumps onto the balanced path at 0. It cannot be the case that both players' normal types have mass points at 0, or else 1, for example, should wait an instant to see if she receives a concession from 2.

Since only one of the probabilities $z_1$and z_2 can jump at 0, and that can only be upward, we see that 1 must have the mass point if an increase in z_1 is required to reach the balanced path (and conversely if z_2 needs to increase). Specifically, if initial priors are z_1 and z_2 and 1 is "weak", then player 1's reputation needs to jump to $z_1^+ = (z_2)^{1/\gamma} > z_1$, in order for the pair $\left(z_1^+, z_2^0\right)$ to lie on the balanced path. By Bayes' rule, conditional upon *not* conceding player 1's posterior reputation equals $\frac{(1-H_1(0))z_1}{1-F_1(0)}$. As we noted in Section 2, $1 - H_1(t) = (1 - F_1(t))^{\alpha_1}$ where $F(t)$ and $H(t)$ are the distribution functions over concession times for player i (averaged over normal and behavioral types), and for a behavioral type of player i, respectively. Let $\omega_1 \equiv F_1(0)$. Then the preceding formula yields 1's posterior reputation $\frac{(1-\omega_1)^{\alpha_1} z_1}{1-\omega_1}$, which must equal $z_1^+ = (z_2)^{1/\gamma}$. This equality implies $1 - \omega_1 = \left(\frac{z_1}{z_1^+}\right)^{1/1-\alpha_1}$. Hence, 1 concedes at 0 with probability

$$\omega_1 = 1 - \left(\frac{z_1}{(z_2)^{1/\gamma}}\right)^{\frac{1}{1-\alpha_1}}.$$

Section 4

Consider any subgame which begins in the interior of the box $O\alpha E\alpha^2$. After the start of the subgame, lumpy concession by either player is impossible. We study paths of continuous concession in this region, keeping in mind that, in equilibrium, after possibly an initial jump, play must evolve along paths. The discussion in section 4 implies that if the priors lie in $0\alpha E\alpha^2$, then any equilibrium path of posterior probabilities must exit this region through point E. Any other exit point *must* result in a jump by the player who is weak to point E and we have ruled out jumps except at the beginning of a subgame.

The following lemmas are useful.

Lemma 4.1 *After any history h, if player i concedes with positive probability in equilibrium at some instant t, then player i's concession must end the game.*

Proof. Let z_i be player i's probability of being behavioral after history h, but prior to an equilibrium concession with positive probability ω_i. It follows from (7) that conditional upon not conceding, player i's posterior probability of being behavioral is

$$z_i^+ = \frac{(1-\omega_i)^{\alpha_i} z_i}{(1-\omega_i)} > z_i > z_i^-$$

where z_i^- is player i's posterior reputation conditional upon conceding. If player i's equilibrium concession is only one unit, and hence does not end the game, player i cannot be *strictly* weak in the full concession game that remains. This is a consequence of the "discrete limit requirement" (see footnote 10). It follows that a profitable deviation for player i is to wait for a moment $\varepsilon > 0$, and then concede one unit, entering the final subgame with reputation $\alpha z_i^+ > z_i^-$; since player i is strong in this final subgame this deviation yields a strictly higher payoff. ■

Lemma 4.2. *If player i concedes in equilibrium with positive probability at some time $t > 0$ after the start of the subgame, then player $j \neq i$ concedes with zero probability in some $\varepsilon > 0$ time interval $(t-\varepsilon, t] \subseteq (0, t]$.*

Proof. For small enough $\epsilon > 0$, conditional upon both players not conceding prior to $t - \epsilon$, the payoff to j of conceding at $t + \epsilon$ exceeds

$$\omega_i x_j + (1-\omega_i)[\widehat{\omega}_i(x_j - 1) + (1-\widehat{\omega}_i)(x_j - 2)] - O(\epsilon) \tag{5}$$

where $\widehat{\omega}_i$ is the probability of lumpy concession immediately following a one unit concession by j at $(t + \epsilon)$.

On the other hand the payoff to j of conceding at any $s \in (t-\epsilon, t]$ is at most

$$\widetilde{\omega}_i(x_j - 1) + (1-\widetilde{\omega}_i)(x_j - 2) + O(\epsilon) \tag{6}$$

where $\widetilde{\omega}_i$ satisfies

$$\frac{(1-\omega_i)(1-\widehat{\omega}_i)^{\alpha}}{(1-\omega_i)(1-\widehat{\omega}_i)} z_i = \frac{(1-\widetilde{\omega}_i)^{\alpha}}{(1-\widetilde{\omega}_i)} z_i \equiv z_i^+$$

The LHS is (up to $O(\epsilon)$) the posterior probability that i is behavioral if i does not concede at t and does not concede at $(t+\epsilon)$ immediately following j's one unit concession at $(t+\epsilon)$. The RHS is player i's posterior probability conditional upon not conceding following upon a one unit concession by j at $s \in (t-\epsilon, t]$. And z_i^+ is the uniquely required posterior probability for players i and j to be on this balanced path of the (full-concession) $(x_j, x_j - 1)-$ subgame. It is easy to verify that $\widetilde{\omega}_i = \omega_i + \widehat{\omega}_i(1-\omega_i)$, and consequently (5) > (6) for small enough $\epsilon > 0$. ■

Lemma 4.3. *Positive probability concession can occur in equilibrium only at the beginning of a subgame.*
Proof. Follows directly from the preceding two lemmas. ■

We first consider region R_{11}, in which both players concede only partially. The analysis of behavior in R_{11} is quite involved, because concession by player i leads to further concession by player j and the probability of the latter depends upon both players' reputations, which are constantly evolving. Hence the trade-off between conceding now and conceding later is far more complicated than in the full concession case of Section 3.

Consider $\left(z_1^0, z_2^0\right) \in R_{11}$. Let (F_1, F_2) define an equilibrium. Let $z_i(t)$ be player i's posterior probability of being behavioral given z_i^0 and F_i, if neither player has conceded until time t. Then by Bayes' rule,[12]

$$z_i(t) = \frac{(1-F_i(t))^{\alpha_i} z_i^0}{1-F_i(t)} \tag{7}$$

Define $\tau_{11} = \inf\{t \mid z(t) \notin R_{11}\}$. Let $u_i(t)$ be player i's utility from conceding at t, given F_j. Let

$$A_i = \left\{ t \mid u_i(t) = \max_{s \geq 0} u_i(s) \right\}.$$

We argue below that

(i) F_1, F_2 are strictly increasing on $[0, \tau_{11}]$.

(ii) $[0, \tau_{11}] \subseteq A_i, \quad i = 1, 2$. It follows that,

(iii) u_i is differentiable on $[0, \tau_{11}], \quad i = 1, 2$. Consequently,

(iv) F_i is differentiable on $[0, \tau_{11}], \quad i = 1, 2$.

Regarding (iv), no qualification for $t = 0$ is required. In the interior of region R_{11} full concession is *not* optimal; this precludes lumpy concession. This is an implication of the previous Lemmas.

[12]Let A be the event that player i does not concede till time t and B be the event that player i is behavioral. Then $P(B|A) = z_i(t) = \frac{P(A|B)P(B)}{P(A)} = \frac{(1-F_i(t))^{\alpha_i} z_i^0}{1-F_i(t)}$

Step 1. F_1, F_2 are continuous on $(0, \tau'')$.

Proof. By definition, in R_{11} the optimal amount of concession is exactly one unit for both players. The result now follows directly from Lemma 4.1. ■

Step 2. u_i is continuous on $(0, \tau'']$.

Proof. Recall the expression for u_i in the text. Continuity of F_1, F_2 implies that the integral term in the expression for u_i is continuous in t and that the reputation pair $(z_1(t), z_2(t))$ evolves continuously in t. The result then follows directly. ■

Steps 3 - 7 establish that *both* F_1 and F_2 are strictly increasing on $[0, \tau'']$.

Step 3. For any t', t'' such that $\tau'' > t'' > t' \geq 0$, either $F_1(t'') > F_1(t')$ or $F_2(t'') > F_2(t')$ (or both).

Proof. Suppose not and let

$$\hat{t} = \inf \left\{ t \mid F_1(t) > F_1\left(t'\right) \text{ or } F_2(t) > F_2\left(t'\right) \right\}$$

Suppose player i satisfies $F_i(t) > F_i(t')$ for all $t > \hat{t}$. Then by continuity of F_i, $z_i\left(\hat{t}\right) = z_i(t')$; absent concession by j until t', the payoff to player i from conceding at $\hat{t}$ is identical to her payoff from conceding at t' except that the latter payoff is received earlier. Hence $u_i(t') > u_i\left(\hat{t}\right)$., By continuity of u_i, $u_i(t) < u_i(t')$ in a neighborhood of $\hat{t}$. But this contradicts the optimality of conceding in a neighborhood of $\hat{t}$, contradicting $F_i(t) > F_i(t')$ for all $t > \hat{t}$. ■

Step 4. Suppose for some $\bar{t} > \underline{t} \geq 0$, $F_1(\bar{t}) = F_1(\underline{t})$ and $F_1(t) > F_1(t')$ for all $t > \bar{t}$. Then (by the previous step) $F_2(t'') > F_2(t')$ all t', t'' such that $\bar{t} > t'' > t' \geq \underline{t}$.

Step 5. Let $\underline{t}$, $\bar{t}$ be as defined in Step 4. As in Steps 6 and 7 of the proof of Proposition 3.1, we may argue that

a) $(\underline{t}, \bar{t}] \subseteq A_2 \equiv \{t \mid u_2(t) = \max_{s \geq 0} u_2(s)\}$

b) $u_2(\cdot)$ is maximal, hence constant, hence differentiable on $[\underline{t}, \bar{t}]$

c) F_2 is therefore differentiable on this interval also; so too is F_1, by virtue of its constancy. It follows that $z_1(t), z_2(t)$ and ω_1, the instantaneous probability with which player 1 concedes to player 2 following a concession of one unit by player 2, are differentiable in t.

Step 6. For $\underline{t} \leq t \leq \bar{t}$, (where $\underline{t}, \bar{t}$ are as in Step 4) consider

$$u_2(t) = \int_0^{\underline{t}} 2e^{-rs} dF_1(s) + \int_{\underline{t}}^{t} 2e^{-rs} dF_1(s) + e^{-rt}(1 - F_1(t))\left[1 + \omega_1(z_1(t), \alpha z_2(t))\right]$$

Since $F_1(\bar{t}) = F_1(\underline{t})$, the second term drops out.

Setting $u_2'(t) = 0$ yields

$$\lambda_2(t) = \frac{f_2(t)}{1 - F_2(t)} = \frac{3(1+\omega_1)r}{1-\omega_1}.$$

This term yields the rate of continuous *solo* concession by player 2. We find below that this rate of concession does not serve to induce player 1 to delay conceding between $\underline{t}$ and $\bar{t}$.

Since

$$u_1(t) = \int_0^{\underline{t}} 3e^{-rs} dF_2(s) + \int_{\underline{t}}^{t} 3e^{-rs} dF_2(s) + e^{-rt}(1 - F_2(t))[2 + \omega_2(\alpha z_1(t), z_2(t))]$$

Hence,

$$u_1'(t) = e^{-rt}(1 - F_2(t))\{\lambda_2(t)(1-\omega_2) - r(2+\omega_2) + \lambda_1(t)(1-\omega_2) - \lambda_2(t)(1-\omega_2)\}$$

where the last two terms reflect

$$\frac{d\omega_2}{dt} = \frac{\partial \omega_2}{\partial z_1}\dot{z}_1 + \alpha\frac{\partial \omega_2}{\partial z_2}\dot{z}_2 = \lambda_1(t)(1-\omega_2) - \lambda_2(t)(1-\omega_2).$$

Since $\lambda_1(t) = 0$,

$$u_1'(t) = e^{-rt}(1 - F_2(t))(-r(2+\omega_2)) < 0$$

Hence, $u_1(\bar{t}) < u_1(\underline{t})$, which together with continuity of u_1 contradicts $F_1(t) > F_1(\underline{t})$ all $t > \bar{t}$, as required by the definition of $\bar{t}$.

Step 7. Repeating Steps 4-6 with the roles of players 1 and 2 interchanged establishes that F_2 is strictly increasing.

Step 8. The preceding steps establish that both F_1 and F_2 are strictly increasing on $[0, \tau_{11}]$. Repeating and adapting by now familiar arguments, it also follows that for $i = 1, 2$, $A_i = [0, \tau_{11}]$, u_i is constant, hence differentiable on $[0, \tau_{11}]$ and consequently so also are F_j, $j = 1, 2$.

Now, if player 2 concedes, she concedes by *one* unit and is *strong* in the subgame (4,2) which results after the concession. If initial reputations in the (4,2) subgame are (z_1, z_2) then player 1 concedes to a strong player 2 with probability $\omega_1(z_1, z_2) = 1 - \left(\frac{z_1}{(z_2)^{1/3}}\right)^{1/1-\alpha}$. Player 1's utility after a concession by 2 is therefore 3 units, her "residual" payoff in the (4,2) subgame. Player 2's post-concession utility, on the other hand, is 2 units if player 1 concedes immediately after player 2 does and 1 unit otherwise, where the latter is player 2's payoff along the balanced path of the (4,2) subgame. The relevant expression is:

$$2\omega_1(z_1(t), \alpha z_2(t)) + 1(1-\omega_1(z_1(t), \alpha z_2(t))) = 1 + \omega_1(z_1(t), \alpha z_2(t))$$

where $(z_1(t), z_2(t))$ are the reputations of the two players just prior to the concession by player 2 and $(z_1(t), \alpha z_2(t))$ are the post-concession reputations.

Conversely, if player 1 concedes, she too concedes by one unit and is the strong player in the resulting (3,3) subgame. Player 2's utility after a concession by 1 is therefore 2 units and player 1's is $2 + \omega_2(\alpha z_1(t), z_2(t))$ where $(z_1(t), z_2(t))$ are pre-concession reputations and ω_2 is the probability with which player 2 concedes to a strong player 1 in the (3,3) subgame

$$\omega_2(z_1, z_2) = 1 - \left(\frac{z_2}{z_1}\right)^{1/1-\alpha}$$

Thus, using various facts noted above

$$u_1(t) = \int_0^t 3e^{-rs} dF_2(s) + e^{-rt}(1-F_2(t))[2+\omega_2(\alpha z_1(t), z_2(t))] \qquad t \in [0, \tau_{11}]$$

Furthermore, by (ii) and (iii), $u_1'(t) = 0$, all $t \in [0, \tau_{11}]$.

Note that

$$\begin{aligned}\frac{d\omega_1(z_1(t), \alpha z_2(t))}{dt} &= \tfrac{\partial \omega_1}{\partial z_1}\dot{z}_1 + \alpha \tfrac{\partial \omega_1}{\partial z_2}\cdot \dot{z}_2 \\ &= -\lambda_1(1-\omega_1(z_1(t), \alpha z_2(t))) + \tfrac{\lambda_2}{3}(1-\omega_1(z_1(t), \alpha z_2(t))).\end{aligned}$$

and

$$\frac{d\omega_2(\alpha z_1(t), z_2(t))}{dt} = (-\lambda_2 + \lambda_1)(1-\omega_2(\alpha z_1(t), z_2(t))).$$

Here we have used the differentiability of F_1, F_2 and the equation

$$\dot{z}_i(t) = (1-\alpha)\lambda_i(t) z_i(t) \tag{8}$$

The latter follows directly by taking logs and differentiating the formula (7)

$$z_i(t) = \frac{(1-F_i(t))^{\alpha_i} z_i^0}{(1-F_i(t))}$$

introduced above.

Differentiating $u_1(t)$ and setting its derivative to zero yields

$$\lambda_1(t) = \frac{f_1(t)}{1-F_1(t)} = \frac{(2+\omega_2)r}{(1-\omega_2)}$$

where for simplicity we suppress the arguments of ω_2.

Also,

$$u_2(t) = \int_0^t 2e^{-rs} dF_1(s) + e^{-rt}(1 - F_1(t))[1 + \omega_1(z_1(t), \alpha z_2(t))].$$

Setting $u_2'(t) = 0$ for $t \in [0, \tau_{11}]$ yields

$$\lambda_2(t) = \frac{f_2(t)}{1 - F_2(t)} = \frac{3(1 + \omega_1) r}{(1 - \omega_1)}.$$

We seek to characterize the path traced by $(z_1(t), z_2(t))$, $t \in [0, \tau_{11}]$ conditional upon neither player conceding until τ_{11}.

By (8) the slope of this path is

$$\frac{dz_2(z_1(t))}{dz_1} = \frac{\dot{z}_2(t)}{\dot{z}_1(t)} = \frac{\lambda_2(t) z_2(t)}{\lambda_1(t) z_1(t)}.$$

For $(z_1, z_2) \in \{(z_1(t), z_2(t)) \mid t \in [0, \tau_{11}]\}$, direct calculation yields

$$\frac{dz_2}{dz_1} = \frac{3z_2 z_2^{\frac{1}{1-\alpha}}}{z_1 z_1^{\frac{1}{1-\alpha}}} \cdot \frac{2(\alpha z_2)^{\frac{1}{3(1-\alpha)}} - z_1^{\frac{1}{1-\alpha}}}{3(\alpha z_1)^{\frac{1}{1-\alpha}} - z_2^{\frac{1}{1-\alpha}}}.$$

The general form of the phase diagram is shown in Figure 7 above.

The salient features are that only a single path reaches point E; this path starts at the origin. All paths above this one hit the αz_1 boundary and have a *steeper* slope at the point of intersection; all paths below hit the z_1^3/α boundary and have slope *flatter* than the lower boundary at the relevant point of intersection. If the initial priors lie on the single path from the origin to E (denote this path (OE)) then there is an equilibrium in which, conditional upon *not* conceding, the posteriors evolve along this path to point E and follow the path defined by $z_2 = z_1^2$ in region R_{22}. Recall that the latter path is the unique "full concession" path in the subgame (4,3). Of course, in region R_{22}, full concession is uniquely optimal and the full concession path is the relevant one, though less than full concession is feasible.

What if priors do *not* lie on (OE)? Then as discussed above, the equilibrium paths lead outside the region R_{11} and it is necessary to analyze equilibrium behavior in regions R_{21} and R_{12} respectively.

Recall that in region R_{21}, player 1's optimal concession is *two* units and player 2's is *one* unit. Proceeding as in our analysis of R_{11} above, we may define

$$\tau_{21} = \inf\{t \mid z(t) \notin R_{21}\}.$$

Points (i) - (iv) continue to be valid in R_{21}. Now, for a subgame which begins in this region, *absent lumpy concession at* $t = 0$, for $t \in [0, \tau_{21}]$

$$u_1(t) = \int_0^t 3e^{-rs} dF_2(s) + e^{-rt} 2(1 - F_2(t))$$
$$u_2(t) = \int_0^t 3e^{-rs} dF_1(t) + e^{-rt}(1 - F_1(t))(1 + \omega_1(z_1(t), \alpha z_2(t)))$$

Setting $u_1'(t) = 0$ yields

$$\frac{f_2(t)}{1 - F_2(t)} = \lambda_2(t) = 2r.$$

Since player 1's utility from conceding is always 2 and her utility when player 2 concedes to her is always 3, the required likelihood ratio $\lambda_2(t)$ is independent of z.

This is not the case for player 2 and $\lambda_1(t)$.

$$u_2'(t) = 0 \quad \Rightarrow \quad (1 + \omega_1) r = \lambda_1(t) + \frac{\lambda_2}{3}(1 - \omega_1) \quad \Rightarrow \quad \lambda_1(t) = \frac{(1 + 5\omega_1) r}{3}$$

Again we have $\frac{dz_2}{dz_1} = \frac{\dot{z}_2}{\dot{z}_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1}$, which now yields

$$\frac{dz_2}{dz_1} = \frac{2z_2}{z_1\left(2 - \frac{5}{3}\left(\frac{z_1}{(\alpha z_2)^{1/3}}\right)^{1/1-\alpha}\right)}$$

The phase diagram for this case is sketched in Figure 8. We first consider paths generated by continuous simultaneous concession by both players.

The important point is that no continuous path of simultaneous concession by both players which starts at some point in R_{21} different from E, ever reaches point E. Finally we consider the possibility of lumpy concession and solo continuous concessions by either player.

Differentiability of u_i, F_i etc., $i = 1, 2$ can be established by repetition or minor adaptations of the arguments presented to prove the analogous results in R_{11}. In the same manner we can show that for any t', t'' such that $\tau_{21} \geq t'' > t' \geq 0$, either $F_1(t'') > F_1(t')$ or $F_2(t'') > F_2(t')$. [See Step 3 above.]

We will argue below that *both* F_1 and F_2 are strictly increasing in this range.

By Lemmas 4.1 to 4.3 lumpy concession is possible only at the *start* of a subgame originating in R_{21}, and such a concession can only be made by player 1. We now consider continuous behavior after time zero.

Suppose the hypothesis in Step 4 above. Then proceeding as in Steps 5 and 6, for u_1 and u_2 *as defined in this region*,

$$u_1'(t) = e^{-rt}\{\lambda_2(t) - 2r\}(1 - F_2(t))$$

It is evident from this expression that solo non-lumpy concession by player 1 (that is, $\lambda_1(t) > 0$ and $\lambda_2(t) = 0$) is impossible: this would require $u_1'(t) = 0$, but $\lambda_2(t) = 0$ implies $u_1'(t) < 0$.

What about the possibility of solo continuous concession by player 2? The requirement that $u_2'(t) = 0$ (implied by the hypothesis that F_2 is strictly increasing on $[\underline{t}, \bar{t}]$) yields

$$(1 + \omega_1) r = \frac{\lambda_2(t)}{3} (1 - \omega_1)$$

(Since $\lambda_1(t) = 0$ by hypothesis)

Hence $u_1'(t) > 0$ for all $t \in [\underline{t}, \bar{t}]$, which implies that there exists $\varepsilon > 0$ such that $u_1(t) < u_1(\bar{t})$ for all $t \in [\underline{t} - \varepsilon, \underline{t}]$ *unless* $\underline{t} = 0$. Thus $\underline{t} > 0$ yields a contradiction (that is, $F_1(\underline{t} - \varepsilon) = F_1(\bar{t})$, contradicting the definition of $\underline{t}$).

Hence the only possibility for solo concession by player 2 is at the *beginning* of the subgame.

As explained earlier, no reputation paths in R_{21} associated with continuous concession by *both* players lead to point E; it is also now obvious that no other path consisting of phases of solo and joint mixing will lead to point E, since the only solo mixing possible is by player 2 and that too only at the *beginning* of the subgame.

For region R_{12} (see Figure 13) we obtain:

$$u_2(t) = \int_0^t 2e^{-rs} dF_1(t) + e^{-rt}(1 - F_1(t))\,1$$

If both players concede continuously it is necessary that:

$$u_2'(t) = 0 \quad \Rightarrow \quad r = \lambda_1(t)$$

and

$$u_1(t) = \int_0^t 4e^{-rs} dF_2(s) + e^{-rt}(1 - F_2(t))\,(2 + \omega_2(\alpha z_1(t), z_2(t)))$$

$$u_1'(t) = 0 \quad \Rightarrow \quad (2 + \omega_2) r = \lambda_2(t) + \lambda_1(t)(1 - \omega_2)$$

for all $t \in [0, \tau_{12}]$ where $\tau_{12} = \inf\{t \mid z(t) \notin R_{12}\}$.

All this yields

$$\frac{dz_2}{dz_1} = \frac{z_2\left(3 - 2\left(\frac{z_2}{\alpha z_1}\right)^{1/1-\alpha}\right)}{z_1}$$

Finally, the only possibility for solo continuous concession is by Player 1 and must occur at the beginning of the subgame. As with region R_{21}, appending an initial phase of solo concession by 1 does not yield a path that reaches point E.

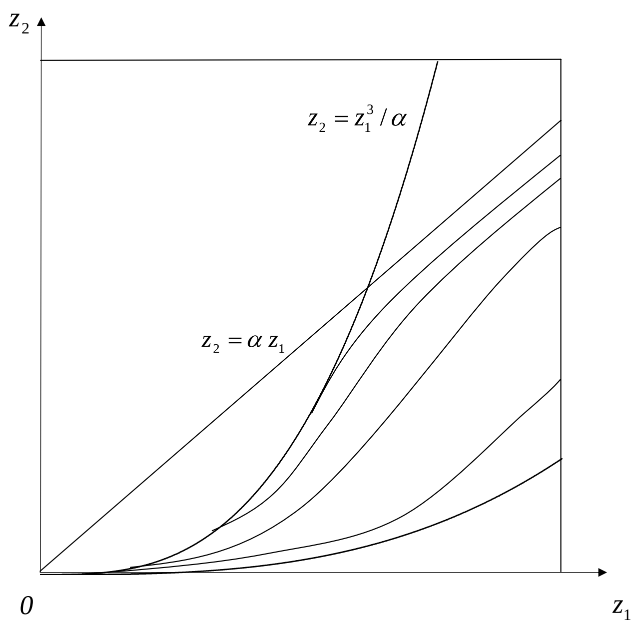

Figure 13: Phase diagram in the region R_{12}.

To summarize, in the rectangle $O\alpha E\alpha^2$, we have found only one balanced path to E, the only way to exit the rectangle in equilibrium. But from a point in $O\alpha E\alpha^2$ *not* on that path, it is impossible to jump onto the curve OE, because on the path, both players' payoffs are *nonresidual.* (Recall that in equilibrium, a lumpy concession by player i must involve full concession, and hence, i's payoff is residual. See Lemma 4.3 above. Her willingness to concede fully implies that her payoff in the non-concession eventuality, that is, upon first reaching the balanced path, must also be *residual.*) We seem to have painted ourselves into a corner, as it were.

Escape is afforded by one degree of freedom not yet exploited. Notice that along the upper boundary of R_{11}, player 1 is indifferent between conceding one unit or two units. If 1 randomizes in the correct (evolving) proportions among conceding zero, one or two units, while 2 randomizes between conceding one unit

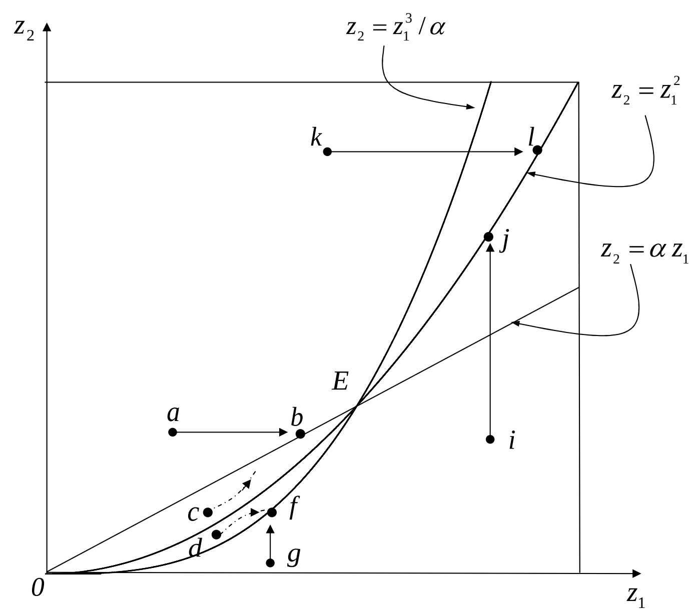

Figure 14: Equilibrium in the $(4, 3)$ subgame.

or none, reputations can move up the αz_1 curve to E, as required. Similarly, 2 can randomize among three possibilities because of the neutrality of her concession preferences along the lower boundary of R_{11}, so play can progress smoothly along that curve as well. These details are worked out immediately following the summary below.

Together, the two boundaries of R_{11} and the path from O to E (denoted by $\widetilde{OE}$) serve as a composite balanced path for the region $O\alpha E\alpha^2$. Figure 14 summarizes the solution. From a point interior to R_{11} and above the curve OE, for example, reputations move up the phase line to αz_1 and then up αz_1 to E. Below the curve OE in R_{11}, reputations move up the (shallower) phase line to the lower boundary of R_{11}, and then follow that boundary up to E. From points to the left of R_{11}, 1 must concede probabilistically, either ending the game or leaving the reputation pair on the upper boundary of R_{11} (where 1's payoff is "residual"). Similarly, from a point below R_{11}, 2 immediately concedes fully, probabilistically, so that if play continues, the reputation pair has jumped vertically to the lower boundary of R_{11}.

As noted earlier, starting from points above the line αz_1 (resp. below the curve z_1^3/α) for $z_i < \alpha$, equilibrium entails reputation evolving along αz_1 (resp. z_1^3/α) up to point E. We show that such an evolution is indeed possible.

Moving along the line αz_1

Along the line αz_1, $\quad \omega_2(\alpha z_1, z_2) = 0$

Though we will not always be explicit, the reader should bear in mind that, in general, λ_1, λ_2, q, etc., are functions of t.

$$\frac{dz_2}{dz_1} = \alpha = \frac{\dot{z}_2}{\dot{z}_1} = \frac{(1-\alpha)\lambda_2 z_2}{(1-\alpha)\lambda_1 z_1} = \frac{\lambda_2}{\lambda_1}\alpha \quad \Rightarrow \lambda_1 = \lambda_2$$

Now, $u_1'(t) = 0 \Rightarrow \lambda_1 = 2r$.

Player 2 obtains expected utility $k(t) = 2q(t) + 3(1 - q(t))$ where $q(t)$ is the probability with which player 1 concedes one unit (conditional upon conceding); she concedes fully (i.e. two units) with complementary probability.

We can sustain a reputation path along αz_1 if and only if we find that $u_2'(t) = 0$ results for some choice of $q(t) \in [0, 1]$.

We confirm that this is indeed the case.

$$\begin{aligned} u_2'(t) = {} & k e^{-rt} f_1(t) - e^{-rt} f_1(t)(1+\omega_1) - r e^{-rt}(1 - F_1(t))(1+\omega_1) \\ & + e^{-rt}(1 - F_1(t))\left[-\lambda_1(1-\omega_1) + \frac{\lambda_2}{3}(1-\omega_1)\right] \end{aligned}$$

Substituting $\lambda_1 = \lambda_2 = 2r$, obtain

$$0 = e^{-rt}(1 - F_1(t))\left[2rk - 2r(1+\omega_1) - r(1+\omega_1) - 2r(1-\omega_1) + \frac{2r}{3}(1-\omega_1)\right]$$

$$3(1+\omega_1) + \frac{4}{3}(1-\omega_1) = 2k$$

$$13 + 5\omega_1 = 6k$$

$$\omega_1 \in [0,1] \Leftrightarrow k \in [2,3] \Leftrightarrow q \in (0,1), \text{ as required.}$$

Moving along the curve $\frac{z_1^3}{\alpha}$

$$\frac{dz_2}{dz_1} = \frac{3z_1^2}{\alpha} = \frac{z_2^0}{z_1^0} = \frac{\lambda_2 z_2}{\lambda_1 z_1} = \frac{\lambda_2 z_1^3}{\lambda_1 z_1 \alpha} \Rightarrow \lambda_2 = 3\lambda_1$$

But $u_2'(t) = 0 \Rightarrow \lambda_2 = \frac{3(1+\omega_1)r}{(1-\omega_1)}$

Along $z_2 = \frac{z_1^3}{\alpha}$, $\omega_1(z_1, \alpha z_2) = 0$. Hence $\lambda_2 = 3r$ and $\lambda_1 = r$.

Now,

$$u_1'(t) = e^{-rt}(1 - F_2(t))\{\lambda_2 h - r(2+\omega_2) - \lambda_2(2 = \omega_2) + \lambda_1(1-\omega_2) - \lambda_2(1-\omega_2)\},$$

where $h = 3q + 4(1-q)$ is player 1's expected payoff when player 2 randomizes between conceding one and two units with probabilities q and $(1-q)$, respectively.

We need to confirm that $h \in [3,4]$, where h satisfies

$$3h - 2 - \omega_2 - 6 + 1 - \omega_2 - 3 = 0$$
$$3h = 10 + 2\omega_2$$
$$\omega_2 \in [0,1] \Rightarrow h \in \left[3\frac{1}{3}, 4\right] \Leftrightarrow q \in [0,1]\,, \text{ as required.}$$

Section 5

It is easiest to identify regions in which player i concedes *fully* in equilibrium. As noted in Lemma 4.1, (probabilistically) lumpy concessions must be full concessions; we focus here on continuous concessions. Consider player 2 and suppose $z_2 < \frac{z_1^3}{\alpha}$. Then even if player 2 concedes two units her post-concession reputation $\alpha z_2 < z_1^3$ is such that she is (strictly) 'weak' in the resultant (4,2) subgame. Were player 2 to concede one unit, then the two players would find themselves in region R_{12} of the (4,3) subgame i.e. a region in which player 2 concedes fully, and in which her payoffs are residual. As argued in footnote 10, player 2 will concede fully in such a situation. An analogous argument applies to player 1 when

$$z_1 < \frac{z_2^3}{\alpha} \qquad \text{i.e. if} \quad z_2 > (\alpha z_1)^{1/3}\,.$$

The preceding argument may be 'reversed'; for $z_2 > \frac{z_1^3}{\alpha}$ it is *not* optimal for player 2 to concede fully. Along the diagonal we have already noted that player 2's optimal concession is *two* units. Fixing z_1 as z_2 increases from the diagonal, it may become optimal for player 2 to concede only *one* unit. From our preceding analysis of the (4,3) and (4,2) subgames there is no difficulty, in principle, in computing player two's payoffs from conceding two and one units respectively and hence identifying the optimal amount of concession.

We seek to derive the boundary L_2, such that above L_2, player 2's optimal concession is one unit only and below L_2 her optimal concession exceeds one unit.

Consider player 2, and suppose $z_2 > \frac{z_1^3}{\alpha}$. If player 2 concedes two units, then her post-concession payoff is

$$2 - \left(\frac{z_1}{(\alpha z_2)^{1/3}}\right)^{1/(1-\alpha)} \tag{9}$$

[Recall that if a player with prior reputation z_1^- concedes lumpily with probability $\omega_1 = 1 - \left(\frac{z^-}{z^+}\right)^{1/1-\alpha}$ (where $z^+ > z^-$) then her posterior reputation, conditional upon not conceding, increases precisely to z_1^+. Player 2's post-concession reputation is αz_2; the balanced path in the (4,2) subgame is $x_2 = x_1^3$, so player 1's reputation needs to jump from z_1 to $(\alpha z_2)^{1/3}$. We may conveniently represent this sequence as follows:

$$(z_1, z_2)\ {}_2\!\downarrow (z_1, \alpha z_2)\ {}_1\!\rightarrow \left((\alpha z_2)^{1/3}, \alpha z_2\right).]$$

Now we compute player 2's payoffs from conceding *one* unit. There are two subcases to consider. If $z_2 > \alpha$, then the posterior reputations in the post-concession (4,2) subgame are in a region where it is optimal for both players to concede fully, the relevant balanced path being $x_2 = x_1^2$. Thus, $(z_1, z_2)\ {}_2\!\downarrow (z_1, \alpha z_2)\ {}_1\!\rightarrow \left((\alpha z_2)^{1/2}, \alpha z_2\right)$, yielding player 2 a utility of

$$3 - 2\left(\frac{z_1}{(\alpha z_2)^{1/2}}\right)^{1/(1-\alpha)} \tag{10}$$

Equating (9) and (10) yields the expression for the boundary L_2 in the region where $z_2 > \alpha$. That is,

$$z_1 = \left(\frac{1}{2\left(\left(\frac{1}{(\alpha z_2)^{1/2}}\right)^{1/(1-\alpha)} - \left(\frac{1}{(\alpha z_2)^{1/3}}\right)^{1/(1-\alpha)}\right)}\right)^{1-\alpha}$$

When $z_2 < \alpha$, following player 1's concession, player 2 concedes lumpily, but conditional upon not conceding in the resultant (4,3) subgame, player 2's optimal concession is only one unit, to which player 1 in turn responds with a lumpy concession of one unit. The sequence is as follows:

$$(z_1, z_2)\ {}_2\!\downarrow (z_1, \alpha z_2)\ {}_1\!\rightarrow (z_2, \alpha z_2)\ {}_2\!\downarrow \left(z_2, \alpha^2 z_2\right)\ {}_1\!\rightarrow \left(\left(\alpha^2 z_2\right)^{1/3}, \alpha^2 z_2\right)$$

and player 2's expected utility may be computed to be

$$\begin{aligned} 3\hat{\omega}_1 + (1-\hat{\omega}_1)\left(2\tilde{\omega}_1 + (1-\tilde{\omega}_1)\right) &= 1 + 2\hat{\omega} + \tilde{\omega} - \hat{\omega}\tilde{\omega} \\ &= 3 - 2\left(\tfrac{z_1}{z_2}\right)^{1/(1-\alpha)} - \left(\tfrac{z_2}{\alpha}\right)^{2/(3-3\alpha)}, \end{aligned} \tag{11}$$

where $\hat{\omega}_1$ (resp. $\tilde{\omega}_1$) is the lumpy probability with which player 1 concedes after player 2's first (resp. second) concession.

Equating (9) and (11) yields the expression for L_2 when $z_2 < \alpha$:

$$z_1 = \left(\frac{1}{\left(\frac{1}{z_2}\right)^{1/(1-\alpha)} + \left(\frac{1}{(\alpha^2 z_2)^{1/3}}\right)^{1/(1-\alpha)} - \left(\frac{1}{(\alpha z_2)^{1/3}}\right)^{1/(1-\alpha)}}\right)^{1-\alpha}$$

Having identified the regions $\hat{R}_{mn}$, we need to verify that equilibrium behavior within these regions is as we have described in Figures 11 and 12.

Consider the region $\hat{R}_{33}$ and the point $(\sqrt{\alpha}, \sqrt{\alpha})$. For any $(z_1, z_2) \in \hat{R}_{33}$ and below the diagonal, the unique equilibrium entails player 2 conceding lumpily so that conditional upon not conceding, her reputation jumps upward to the balanced path $z_2 = z_1$. This fact precludes smooth entry into region $\hat{R}_{33}$ from strictly below. An analogous argument precludes smooth entry from the region strictly to the left of $\hat{R}_{33}$. This implies that unique behavior outside the box $0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$ is as diagrammed in Figures 11 and 12.

This argument is analogous to that in section 4 (the (4,3) subgame) for the area outside the box $0\alpha E\alpha^2$.

Now consider behavior within the box $0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$. We need to derive the line L_2^* (mentioned in Section 5) such that along L_2^*, solo and incentive compatible concession by player 1 (that is, $\lambda_2(t) = 0$ and $\lambda_1(t) > 0$ such that $u_1'(t) = 0$) which continues up to the diagonal is just incentive compatible for player 2. To the left of this line player 2 strictly prefers to concede right away rather than await continuous solo concession by player 1, and to then subsequently concede (in the probabilistic event that player 1 does not concede during her period of solo concession) when player 1's posterior reputation equals z_2 and the reputational pair reaches the diagonal. To the right of this line player 2 strictly prefers to wait. It may be directly checked that solo incentive compatible concession by player 1 yields $u_2' > 0$ in $\hat{R}_{22}$ and $u_2' < 0$ in $\hat{R}_{21}$. It follows that L_2^* lies strictly to the left of L_2 (See Figures 11 & 12). Explicit calculations (we are grateful to Yuliy Sannikov for performing these) reveal that L_2^* starts at the origin, is continuous and strictly increasing, and lies to the right of the boundary $(\alpha z_1)^{1/3}$ until L_2^* terminates at point e on $(\alpha z_1)^{1/3}$, where e lies strictly to the left and below point H, the corresponding point of intersection of L_2 with $(\alpha z_1)^{1/3}$.

We now complete the argument that the behavior described in the text is indeed equilibrium behavior and turn subsequently to the issue of uniqueness.

The important missing element is establishing that movement along the OH segment of the $(\alpha z_1)^{1/3}$ curve is indeed possible. OH lies on the border of $\hat{R}_{31}$ and $\hat{R}_{21}$. Hence, along OH player 1 is indifferent between conceding two and three units; her payoffs are therefore residual. Her utility as a function of t is

$$u_1(t) = \int_0^t 2e^{-rs} dF_2(s) + e^{-rt}(1 - F_2(t))$$

Setting $u_1'(t) = 0$ yields

$$\lambda_2 = r \quad \text{independently} \quad \text{of} \quad t \tag{12}$$

On the other hand,

$$u_2(t) = \int_0^t ke^{-rs} dF_1(s) + e^{-rt}(1 - F_1(t))(1 + 2\omega_1)$$

where $k \in [3,4]$, taking account of the fact that player 1 randomizes between conceding two and three units, respectively.

Hence,

$$u_2'(t) = e^{-rt}(1 - F_1(t))\left\{k\lambda_1 - r(1+2\omega_1) - \lambda_1(1+2\omega_1) \ + \ 2(1-\omega_1)\left(-\lambda_1 + \frac{\lambda_2}{2}\right)\right\}$$

Setting $u_2'(t) = 0$ yields

$$(k-3)\lambda_1 - r(1+2\omega_1) + \lambda_2 - \lambda_2\omega_1 = 0 \tag{13}$$

The slope of the equilibrium path $(z_1(t), z_2(t))$ is

$$\left.\frac{dz_2}{dz_1}\right|_{z(t)} = \frac{\dot{z}_2(t)}{\dot{z}_1(t)} = \frac{(1-\alpha)\lambda_2(t)z_2(t)}{(1-\alpha)\lambda_1(t)z_1(t)} = \frac{\lambda_2 z_2}{\lambda_1 z_1}$$

On the other hand the slope of the OH curve, $z_2 = (\alpha z_1)^{1/3}$, is

$$\frac{dz_2}{dz_1} = \frac{1}{3}(\alpha z_1)^{-2/3}\alpha$$

Equating these slopes and substituting $\lambda_2 = r$ from (12) above yields

$$\lambda_1 = 3r$$

Substituting into (13) and simplifying yields

$$(k-3) = \omega_1,$$

which in turn implies $k \in [3,4]$. Hence appropriate randomization by player 1 leads to an equilibrium path which indeed evolves along OH as required.

Thus, we have established that starting from any reputational pair, there exists an equilibrium as described in the text.

Now we argue that the paths described are *unique*.

We know that

i. Outside the box $0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$, unique equilibrium behavior entails the 'weak' player conceding lumpily so that conditional upon non-concession the reputational pair jumps to the diagonal and thereafter the posteriors evolve along the diagonal with both players conceding three units (i.e. fully) at identical continuous rates.

ii. Lumpy concession can only occur at the beginning of an equilibrium path (Lemma 4.3).

iii. Hence any equilibrium path which originates within the box can only exit the box through the point $\hat{E}$.

We will present the rest of the argument for initial reputational pairs which lie above the diagonal. A symmetric analysis applies for starting points below the diagonal.

From strictly within the box, exit via $\hat{E}$ can only occur in the following ways:

1) along the diagonal

2) from *within* the region $\hat{R}_{22}$, excluding the diagonal itself

3) along the border of $\hat{R}_{32}$ and $\hat{R}_{22}$ $\left(H\hat{E}\right)$ or along the corresponding border $\left(\hat{R}_{23} \cap \hat{R}_{22}\right)$ below the diagonal.

4) from strictly *within* the region $\hat{R}_{32}$ (or $\hat{R}_{23}$)

We will argue that the last three are impossible.

(2) is impossible. As argued in the text, in $\hat{R}_{22}$,

$$\lambda_1 = r\frac{(1+\omega_1) - (1+\omega_2)(1-\omega_1)/3}{1-(1-\omega_1)(1-\omega_2)/9},$$

where

$$\omega_1 = 1 - \left(\frac{z_1}{(\alpha z_2)^{1/3}}\right)^{1/(1-\alpha)}, \text{ and} \lambda_2 \text{ and } \omega_2 \text{ are defined symmetrically.}$$

The slope of the equilibrium path is

$$\frac{dz_2}{dz_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1}.$$

Let $z_2 = bz_1$ and

$$\varphi(b) = \frac{dz_2}{dz_1}|_{z_2 = bz_1}$$

Then clearly $\varphi(1) = 1$. Tedious calculations reveal that

$$\varphi'(b)|_{b=1} > 1$$

Hence $\hat{E}$ and *indeed any point on the diagonal* cannot be approached via a path of continuous concessions by *both* players in $\hat{R}_{22}$.

(3) is impossible.

Recall that $H\hat{E}$ lies on the border of the regions $\hat{R}_{32}$ and $\hat{R}_{22}$.

$$u_1(t) = \int_0^t 3e^{-rs} dF_2(s) + e^{-rt}(1 - F_2(t))$$

Hence $u_1'(t) = 0$ yields $\lambda_2 = \frac{r}{2}$.
Along $H\hat{E}$ $\frac{dz_2}{dz_1} = \frac{1}{3}(\alpha z_1)^{-2/3}\alpha$
Equating the above with the slope of the equilibrium path $\frac{dz_2}{dz_1} = \frac{\lambda_2 z_2}{\lambda_1 z_1}$ yields

$$\lambda_1 = \frac{3r}{2}$$

Now

$$u_2(t) = \int_0^t ke^{-rs}dF_1(s) + e^{-rt}(1 - F_1(t))(1 + \omega_1),$$

where $k \in [3, 4]$, since player 1 may randomize between conceding 2 and 3 units respectively.

Hence $u_2'(t) = 0$ yields

$$k\lambda_1 - r(1 + \omega_1) - \lambda_1(1 + \omega_1) + (1 - \omega) - \left(\lambda_1 + \frac{\lambda_2}{3}\right) = 0$$

Substituting $\lambda_1 = \frac{3r}{2}, \lambda_2 = \frac{r}{2}$ yields

$$3k = \frac{23}{3} + \frac{7}{3}\omega_1 \quad g(k-3) = 7\omega_1 - 4 \quad \Rightarrow 7\omega_1 \geq 4 \qquad \omega_1 \geq \frac{4}{7}$$

But $\omega_1 \to 0$ as $z_1 \to \sqrt{\alpha}$, i.e. near $\hat{E}$.

Thus, close to $\hat{E}$, equilibrium movement along $H\hat{E}$ is impossible: for the equations to be satisfied, player 1 would need to concede three units with *negative* probability.

(4) is impossible.

Along $(\alpha z_1)^{1/3}$, $\frac{dz_2}{dz_1} = \frac{1}{3}(\alpha z_1)^{-2/3}\alpha$. Near $\hat{E}$ this expression approximately equals $\frac{dz_2}{dz_1}|_{z_1=\sqrt{\alpha}} = 1/3$. On the other hand, in $\hat{R}_{32}$ the vector field has slope

$$\frac{dz_2}{dz_1} = \frac{z_2}{z_1}\frac{6}{12 - 7\left(\frac{z_1}{(\alpha z_2)^{1/3}}\right)^{1/1-\alpha}}$$

Near $\hat{E}$ this expression approximately equals

$$\frac{6}{12 - 7\left(\frac{(\alpha^{1/2})^{2/3}}{\alpha^{1/3}}\right)^{1/1-\alpha}} = \frac{6}{5} > 1$$

This implies that approaching $\hat{E}$ from $\hat{R}_{32}$ is impossible, since this would entail an equilibrium path with slope not greater than 1/3 near $\hat{E}$.

Hence the final approach to $\hat{E}$ must be along the diagonal. It remains to argue that starting from any reputational pair there is a unique way to make this final approach. In particular, we need to rule out possibilities illustrated in Figures 15 to 17 below. The bold line indicates the unique equilibrium evolution of reputation and the dotted lines, equilibrium possibilities that we need to rule out.

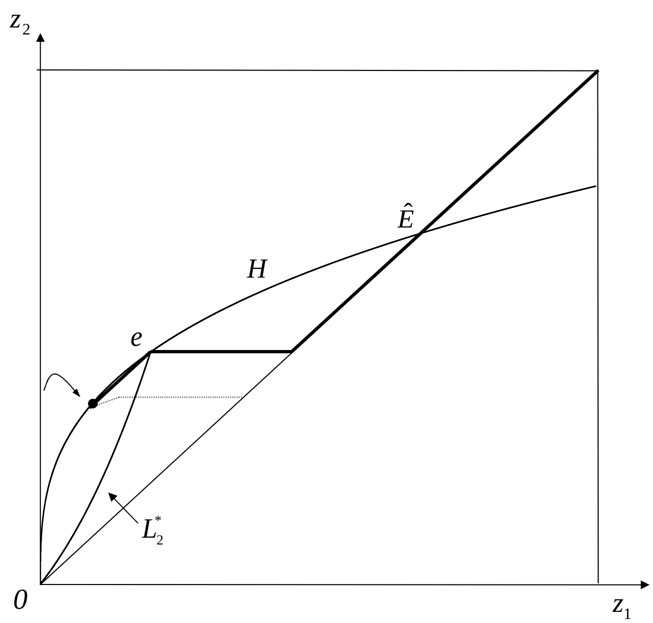

Figure 15: Equilibrium possibility that needs to be ruled out in the $(4, 4)$ subgame.

These issues are addressed in points (5)-(9) below.

(5) From an initial point like c in Figure 11, strictly inside the box and above the curve $(\alpha z_1)^{1/3}$, equilibrium must entail concession by 1 with positive probability so that conditional upon no concession, player 1's reputation jumps from c to d as indicated. This is necessary because in $\hat{R}_{31}$ and $\hat{R}_{32}$, continuous solo concession by player 1 is impossible (1's payoffs are residual) and simultaneous continuous concession yields

$$\frac{dz_2}{dz_1} = \begin{cases} \frac{z_2}{z_1} \frac{1}{3(1-(\frac{z_1}{(\alpha z_2)^{1/2}})^{1/(1-\alpha)})} & \text{in} \quad \hat{R}_{31} \quad \text{when} \quad z_2 > \alpha \\ \frac{z_2}{z_1} \frac{-3}{(2(\frac{z_1}{z_2})^{1/(1-\alpha)}-1)(2(\frac{z_2^{2/3}}{\alpha^{2/3}})^{1/(1-\alpha)}-1)} & \text{in} \quad \hat{R}_{31} \quad \text{when} \quad z_2 < \alpha \\ \frac{z_2}{z_1} \frac{6}{12-7(\frac{z_1}{(\alpha z_2)^{1/3}})^{1/(1-\alpha)}} & \text{in} \quad \hat{R}_{32} \end{cases}$$

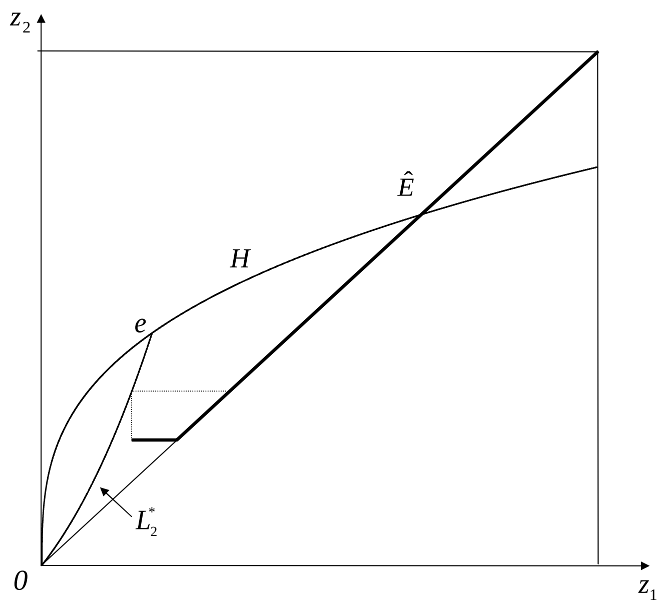

Figure 16: Equilibrium possibility that needs to be ruled out in the $(4, 4)$ subgame.

which in all cases is strictly greater than the slope of $(\alpha z_1)^{1/3}$ for (z_1, z_2) near enough the curve $(\alpha z_1)^{1/3}$.

It follows that once an equilibrium path (of reputational pairs) lies on or below $(\alpha z_1)^{1/3}$, it can never go strictly above this curve (See point (ii) above).

(6) Consider the subregion of $\hat{R}_{21}$ defined by $(\alpha z_1)^{1/3}$ on the left and L_2^* on the right. In this region both players must concede continuously. Furthermore, setting $\lambda_1 = 0$, $u_2' = 0$ yields λ_2 such that $u_1' < 0$. On the other hand, $\lambda_2 = 0$, $u_2' = 0$ yields λ_1 such that $u_2' < 0$. The only possibility is simultaneous continuous concession.

This yields unique equilibrium paths from any initial point which evolve to either the left or right boundaries of this region. It is easy to check that in $\hat{R}_{21}$,

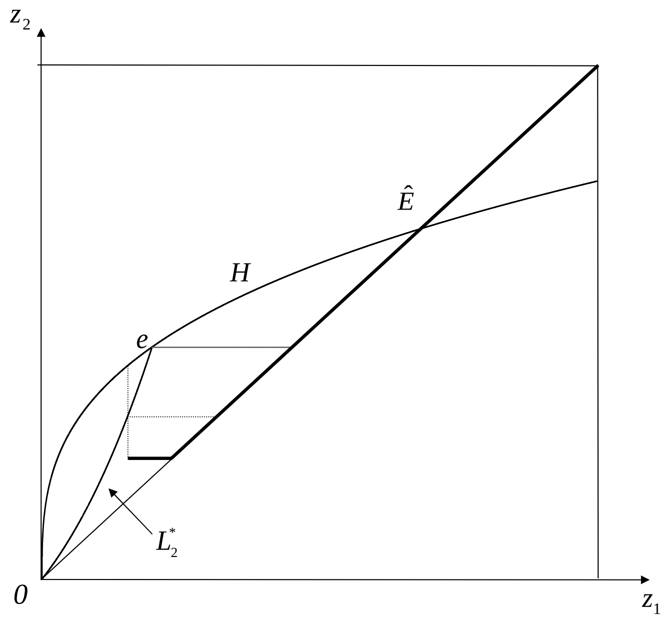

Figure 17: Equilibrium possibility that needs to be ruled out in the $(4,4)$ subgame.

$$\frac{dz_2}{dz_1} = \frac{z_2}{z_1} \cdot \frac{3(\frac{(\alpha z_2)^{1/2}}{z_1})^{1/(1-\alpha)} - 2}{6(\frac{(\alpha z_1)^{1/3}}{z_2})^{1/(1-\alpha)} - 3}$$

when $z_2 > \alpha$ and

$$\frac{dz_2}{dz_1} = \frac{z_2}{z_1} \cdot \frac{\left(3 - \left(\frac{z_1}{z_2}\right)^{1/(1-\alpha)} - \left(\frac{z_1}{(\alpha^2 z_2)^{1/3}}\right)^{1/(1-\alpha)}\right)}{3\left(2\left(\frac{(\alpha z_1)^{1/3}}{z_2}\right)^{1/(1-\alpha)} - 1\right)\left((\frac{z_1}{z_2})^{1/(1-\alpha)} + 1/3(\frac{z_1 z_2^{1/3}}{\alpha^{2/3}})^{1/(1-\alpha)}\right)}$$

when $z_2 < \alpha$.

This slope is steeper than the slope of $(\alpha z_1)^{1/3}$ in a neighborhood of $(\alpha z_1)^{1/3}$ contained in $\hat{R}_{21}$.

This means that starting from a point on $(\alpha z_1)^{1/3}$ (and below c), the only equilibrium possibility is evolution *along* $(\alpha z_1)^{1/3}$ to point e. Furthermore it is the case that $\frac{dz_2}{dz_1}$ is flatter than L_2^* in a neighborhood of L_2^*. This implies that L_2^* cannot be approached from *below*, along an equilibrium path.

(7) The final approach to the diagonal must be via *solo* continuous concession by 1, since by the proof of (2) above, $dz_2/dz_1 > 1$ for simultaneous concession in a neighborhood of the diagonal. That is, the diagonal cannot be approached along a path of *simultaneous* concession in $\hat{R}_{22}$. At any point along the final approach (i.e., the final horizontal segment) to the diagonal and strictly to the *right* of L_2^*, player 2 obtains strictly higher utility from the equilibrium path than from conceding right away. This precludes the final approach from being preceded by a segment of continuous concession, unless the final horizontal segment begins on L_2^*.

(8) No point on L_2^* can be approached from below L_2^*. We have already noted (see (6) above) that *simultaneous* concession in $\hat{R}_{21}$ yields a vector field which is *flatter* than L_2^*. On the other hand, solo concession by 2 is precluded since

$$u_1' = e^{-rt}(1 - F_2(t))(2\lambda_2 - r(1+\omega_2) - \lambda_2(1+\omega_2) + (1-\omega_2)(-\lambda_2 + \lambda_1/3)),$$

which is strictly negative when $\lambda_1 = 0$.

(9) Hence from a point like i to the right of L_2^* and above the diagonal, the only possibility is solo concession by 1 until the diagonal is reached. □

Proof of Proposition 5.1. Recall that for (z, z) with $z < \sqrt{\alpha}$, players optimally concede two units in equilibrium. Player 2's equilibrium expected payoff is:

$$v_2(z, z) = 1 + \omega_1(z, \alpha z) = 2 - \left(\frac{z}{(\alpha z)^{1/3}}\right)^{\frac{1}{1-\alpha}}$$

Clearly, $\lim_{z\to 0} v_2(z, z) = 2$. ■

Proof of Proposition 5.2. Suppose $\lim_{n\to\infty} \frac{z_2^n}{z_1^n} = a > 1$. Then for large n, the reputational pair (z_1^n, z_2^n) lies in region $\hat{R}_{21}$ of the $(4, 4)$ subgame. We recall the players' payoffs in this region from the analysis of Section 5 in this appendix.

Player 1's payoffs in this region are $u_1 = 1 + \omega_2$ when $\omega_2 = 1 - \left(\frac{z}{(\alpha z)^{1/3}}\right)^{\frac{1}{1-\alpha}}$. It follows that $u_1 \to 1$ as $\frac{z_2^n}{z_1^n} \to a > 1$, $z_i^n \to 0$, $i = 1, 2$.

If a is less than the slope of the line L_2^* at the origin, then

$$u_2 = 1 + 2\hat{\omega}_1 + \tilde{\omega}_1 - \hat{\omega}_1\tilde{\omega}_1,$$

where

$$\hat{\omega}_1 = 1 - \left(\frac{z_1}{z_2}\right)^{\frac{1}{1-\alpha}}$$

and

$$\tilde{\omega}_1 = 1 - \left(\frac{z_2}{\alpha}\right)^{\frac{2}{3(1-\alpha)}}$$

It follows that $u_2 \to 3 - \left(\frac{1}{\alpha}\right)^{\frac{1}{1-\alpha}}$.

Finally, if α exceeds the slope of L_2^* at the origin, then

$$u_2 \geq 1 + 2\hat{\omega}_1 + \tilde{\omega}_1 - \hat{\omega}_1\tilde{\omega}_1$$

and the required inequality follows directly. ■

Proof of Proposition 5.3. For large n the reputational pair lies in the region $\hat{R}_{22}$ (see Figures 10 and 11), in which both players' payoffs are residual: they play a full concession game with balanced path given by the diagonal. If player 1 is weak then $u_1 = 1$ and $u_2 = 4\omega_1 + (1 - \omega_1)$, where $\omega_1 = 1 - \left(\frac{z_1}{z_2}\right)^{\frac{1}{1-\alpha}}$. Since $\omega_1 \to 0$, the result follows. ■

Proof of Proposition 5.4. These results may be directly verified by examining the players' payoff functions in the various regions.

Outside $0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$, the result is obvious: At a point like a, player 2's payoff is residual and player 1's payoff equals $4\bar{\omega}_2 + (1 - \bar{\omega}_2) \cdot 1$, where $\omega_2 = 1 - \left(\frac{z_2}{z_1}\right)^{1/(1-\alpha)}$ is the probability with which player 2 concedes immediately. The latter probability is obviously increasing in z_1 and decreasing in z_2. A symmetric argument applies to points outside the box and above the diagonal.

Now consider $(z_1, z_2) \in 0\sqrt{\alpha}\hat{E}\sqrt{\alpha}$. The above discussion clarifies that from all points like c at which one player concedes lumpily (hence has residual payoffs), the other player's payoff (in the case of point c, player 2) is increasing in her own reputation and decreasing in the other players.

For points in $\hat{R}_{21}$ to the left of L_2^*, the equilibrium path entails simultaneous concession by both players. Player 1's payoff is $v_1(z_1, z_2) = 1 + \omega_2$, where

$$\omega_2 = 1 - \left(\frac{z_2}{(\alpha z_1)^{1/3}}\right)^{1/(1-\alpha)}.$$

Player 2's payoff is

$$\begin{cases} v_2(z_1, z_2) = 1 + 2\omega_1 \quad \text{when} \quad z > \alpha, \quad \text{where} \quad \omega_1 = 1 - \left(\frac{z_1}{(\alpha z_2)^{1/2}}\right)^{1/(1-\alpha)} \\ v_2(z_1, z_2) = 1 + 2\hat{\omega}_1 + \tilde{\omega}_1 - \hat{\omega}_1\tilde{\omega}_1 \quad \text{when} \quad z < \alpha, \\ \qquad \text{where} \quad \hat{\omega}_1 = 1 - \left(\frac{z_1}{z_2}\right)^{1/(1-\alpha)} \quad \text{and} \quad \tilde{\omega}_1 = 1 - \left(\frac{z_2^{2/3}}{\alpha^{2/3}}\right)^{1/(1-\alpha)} \end{cases}$$

Differentiating these expressions yields the required result.

We may proceed in a similar way to analyze region $\hat{R}_{22}$.

Finally consider region $\hat{R}_{21}$. The unexpected result is that $\frac{\partial v_2(z_1,z_2)}{\partial z_1} > 0$ in this region. This follows directly from the fact that solo concession by player 1 yields $u_2' < 0$ in region $\hat{R}_{21}$. (Recall that this is compensated by a period with $u_2' > 0$ for solo incentive compatible concession by player 1 between L_2 and the diagonal). ■

Proof of Proposition 5.5. Obvious – players' payoff functions along the diagonal are precisely as indicated. ■

Proof of Proposition 5.6. As $\alpha^n \to 0$, it may be directly checked that region $\hat{R}_{33}$ increases monotonically and approaches the entire unit square. Thus, for any initial (z_1, z_2), eventually players' payoffs are residual, and optimal concessions end the game. That is, normal players either stick with their initial demands of 4 units, or concede completely. ■

Proof of Proposition 5.7. When $\alpha^n \to 1$, region $\hat{R}_{33}$ shrinks to the point (1,1) and player i's optimal concession of 2 units yields i a payoff of

$$\tilde{v}(\alpha^n) = 2 - \left(\frac{z}{(\alpha z)^{1/3}}\right)^{1/(1-\alpha^n)} \to 2$$

as $\alpha^n \to 1$. ■

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