Appendix C. Proofs

**Condition A1. Random draw from population.** Let \( \mu \) be a probability measure on \((\Omega, \mathcal{F})\). Each \( \omega \in \Omega \) represents an individual. \((\Omega, \mathcal{F}, \mu)\) describes the probabilities of drawing individuals from a (possibly infinite) population.

**Condition A2. Stochastic treatment assignment.** For each \( \omega \in \Omega \), let \( v_\omega \) be a probability measure on \((\Delta, \mathcal{D})\). \((\Delta, \mathcal{D}, v_\omega)\) describes the probabilities associated with receiving the treatment (or, in the RDD, the score \( V \)), for each individual \( \omega \). Assume that for any \( B \in \mathcal{D} \), \( v_\omega (B) \) as a function of \( \omega \) is measurable \( \mathcal{F} \). Let \( \mathcal{G} \) be the \( \sigma \)-field consisting of all sets \( \Omega \times A \), where \( A \in \mathcal{D} \).

**Condition A3. Probabilities for the overall experiment.** Define \( P \) as follows: \( \forall E \in \mathcal{F} \times \mathcal{D} \), \( P (E) = \int_{\Omega} v_\omega [\delta : (\omega, \delta) \in E] \mu (d\omega) \). It can be shown that \( P \) is a probability measure on \((\Omega \times \{0, 1\}, \mathcal{F} \times \mathcal{D})\).

**Condition A4. Pre-determined characteristics.** Let \( X = x (\omega) \) be a real-valued function that is measurable \( \mathcal{FD} \). It follows that it is also measurable \( \mathcal{F} \).

**Condition A5. Finite first moments.** \( E_P \) and \( E_\mu \) denote expectations with respect to probability measures \( P \) and \( \mu \), respectively. Where appropriate, \( Y, Y_1, Y_0, \frac{f_\omega (0)}{f_\omega (0)} Y, \frac{f_\omega (0)}{f_\omega (0)} Y_1, \) and \( \frac{f_\omega (0)}{f_\omega (0)} Y_0 \) are each assumed to be integrable \( P \) and integrable \( \mu \).

**Condition B1. Binary treatment assignment model.** Let \( \Delta = \{0, 1\} \) and \( \mathcal{D} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \). Define the random variable \( D \) as \( D = \delta, \delta \in \Delta \), which is measurable \( \mathcal{FD} \).

**Condition B2. Regression discontinuity design.** Let \( \Delta = \mathbb{R} \), and \( \mathcal{D} = \mathbb{R}^1 \) be the class of linear Borel sets. Define the random variable \( V \) – measurable on \( \mathcal{FD} \) – as \( V (\delta) = \delta, \delta \in \Delta \), and let \( D = 1 [V \geq 0] \).

**Condition C1. Potential outcomes.** Let \( Y_1 = y_1 (\omega), Y_0 = y_0 (\omega) \), be real-valued functions that are measurable \( \mathcal{FD} \) (and hence measurable \( \mathcal{F} \)). Let \( Y = DY_1 + (1 - D) Y_0 \).

**Condition C2. Potential outcome function.** Let \( Y = y (\omega, \delta) \) be a real-valued function that is measurable \( \mathcal{FD} \). Let \( y (\cdot, \cdot) \) be continuous in the second argument except at \( \delta = 0 \), where the function is only continuous from the right. Define the function \( Y^+ = y (\omega, 0) \) and \( Y^- = \lim_{\varepsilon \to 0^+} y (\omega, -\varepsilon) \).
**Condition D1. Treatment randomization.** $v_{\omega}$ is identical for all $\omega \in \Omega$

**Condition D2. Continuous density of score.** Let $F_{\omega} (\delta) = v_{\omega} (-\infty, \delta]$, and $f_{\omega} (\delta)$ its derivative with respect to $\delta$. Let $f (\delta) = \int_{\Omega} f_{\omega} (\delta) \mu (d\omega)$. Assume that $0 < f_{\omega} (\delta)$, and $f_{\omega} (\delta)$ is continuous in $\delta$ on $\mathbb{R}$. (Note that if $v_{\omega}$ is measurable $\mathcal{F}$, one can show that in this set-up, so too are $F_{\omega}$ and $f_{\omega}$).

**Proposition 1.** If Conditions A1-A5, B1, C1, and D1 hold, then:

a) $\forall F \in \mathcal{F}$, $P [F \times \Delta | D = 1] = P [F \times \Delta | D = 0] = P [F \times \Delta] = \mu [F]$  

b) $E_{P} [Y | D = 1] - E_{P} [Y | D = 0] = E_{\mu} [Y_{1} - Y_{0}] \equiv ATE$  

c) $\forall x_{0} \in \mathbb{R}, P [X \leq x_{0} | D = 1] = P [X \leq x_{0} | D = 0] = P [X \leq x_{0}] = \mu [\omega : X \leq x_{0}]$  

**Proof.** a) $P [F \times \Delta | D = 1] = P [(F \times \Delta) \cap (\Omega \times \{1\})] / P [\Omega \times \{1\}]$. Numerator is $\int_{F \times \{1\}} P (d (\omega, \delta))$. This is equal to $\int_{F} \left[ \int_{\Omega} v_{\omega} (d\delta) \right] \mu (d\omega) = v_{\omega} (\{1\}) \cdot \mu [F]$ by 18.20.c of Billingsley (1995) and by D1. Similarly, denominator is $v_{\omega} (\{1\})$. Similar argument holds for $P [F \times \Delta | D = 0]$. b) Need to show that conditional expectation of $Y_{1}$ given $\mathcal{G}$, evaluated at $D = 1$ is equal to $E_{\mu} [Y_{1}]$. It can be shown that the conditional expectation of $Y_{1}$ given $\mathcal{G}$ can be written as $\alpha (\delta_{0}) \equiv \frac{1}{P [\Omega \times \{1\}]} \int_{\Omega \times \{1\}} Y_{\delta_{0}} P (d (\omega, \delta))$, for $\delta_{0} = 0$ and 1. Consider the case when $\delta_{0} = 1$. We then have $rac{1}{P [\Omega \times \{1\}]} \int_{\Omega \times \{1\}} Y_{1} P (d (\omega, \delta)) = \frac{1}{P [\Omega \times \{1\}]} \int_{\Omega} \left[ \int_{\Omega} Y_{1} v_{\omega} (d\delta) \right] \mu (d\omega)$ by 18.20.c of Billingsley (1995). Because $Y_{1}$ is only a function of $\omega$, and by D1, this becomes $\frac{1}{P [\Omega \times \{1\}]} \int_{\Omega} Y_{1} \mu (d\omega)$ which is equal to $\int_{\Omega} Y_{1} \mu (d\omega) = E_{\mu} [Y_{1}]$; a similar argument shows that $\alpha (0) = E_{\mu} [Y_{0}]$. c) By A4, for every $x_{0} \in \mathbb{R}$, $F \equiv [\omega : X (\omega) \leq x_{0}]$ is in $\mathcal{F}$, and thus c) follows from a).

**Proposition 2** If Conditions A1-A5, B2, C1, and D2 hold, then:

a) $\forall F \in \mathcal{F}$, $P [F \times \Delta | V = v]$ is continuous in $v$ at $v = 0$  

b) $E_{P} [Y | V = 0] - \lim_{v \to 0^{+}} E_{P} [Y | V = -v] = E_{P} [Y_{1} - Y_{0} | V = 0] = E_{\mu} \left[ \frac{f_{\omega} (0)}{f (0)} (Y_{1} - Y_{0}) \right] \equiv ATE^{*}$  

c) $\forall x_{0} \in \mathbb{R}, P [X \leq x_{0} | V = v]$ is continuous in $v$ at $v = 0$  

**Proof.** a) Fix $F \in \mathcal{F}$, and consider the function $\alpha : \Omega \times \Delta \to \mathbb{R}$, $\alpha (z, \delta) \equiv \frac{f_{\omega} f_{\omega} (\delta) \mu (d\omega)}{f (\delta)}$. It suffices to show 1) that $\alpha (z, \delta)$ is a version of the conditional probability of $F \times \Delta$ given $\mathcal{G}$, and 2) that $\alpha (z, \delta)$ is continuous in $\delta$ on $\mathbb{R}$. First, for each $\Omega \times A$ we have – by 18.20.c and 18.20.d of Billingsley
(1995) – \( \int_{\Omega \times A} \alpha (z, \delta) P (d (z, \delta)) = \int_A \frac{\int f \cdot f_0 (\delta) \mu (d \omega)}{f (\delta)} v (d \delta) \), where \( v \) is a probability measure defined by \( v (B) = \int_{\Omega} v_\omega (B) \mu (d \omega) \), for all \( B \in \mathcal{D} \). \( v \) has density \( f \) with respect to Lebesgue measure because for all \( B \in \mathcal{D} \), \( \int_B f (\delta) \, d \delta = \int_B [\int f_\omega (\delta) \mu (d \omega)] \, d \delta = \int_{\Omega} [\int_B f_\omega (\delta) \, d \delta] \mu (d \omega) = \int_{\Omega} v_\omega (B) \mu (d \omega) \), by Fubini’s theorem, and because \( f_\omega (\delta) \) is a density of \( v_\omega \). Thus, by theorem 16.11 of Billingsley (1995), 
\( \int_A \frac{\int f \cdot f_0 (\delta) \mu (d \omega)}{f (\delta)} v (d \delta) = \int_A [\int f \cdot f_\omega (\delta) \mu (d \omega)] \, d \delta = \int \int_{\Omega} f \cdot f_\omega (\delta) \, d \delta \mu (d \omega) \), by Fubini’s theorem. This equals \( \int f \cdot v_\omega (A) \mu (d \omega) = P [F \times A] \), because \( f_\omega \) is a density and by 18.20.c of Billingsley (1995).

Second, to show continuity of \( \alpha (z, \delta) \), it suffices to show that for any \( F \in \mathcal{F} \) and any sequence \( \delta_n \to 0 \), \( \int f \cdot f_\omega (\delta_n) \mu (d \omega) \to \int f \cdot f_\omega (0) \mu (d \omega) \). This follows from dominated convergence, noting that \( f_\omega (\delta_n) \leq g_\omega \), if \( g_\omega \equiv \sup_n f_\omega (\delta_n) \), which is finite for each \( \omega \), because \( f_\omega (\delta_n) \) converges to \( f_\omega (0) \), by D2.

b) Consider the function \( \beta : \Omega \times \Delta \to \mathbb{R} \), \( \beta (z, \delta) = \int_{\Omega} Y \frac{f_\omega (\delta)}{f (\delta)} \mu (d \omega) \). It suffices to show that

1) \( \beta (z, \delta) \) is a version of the conditional expectation of \( Y \) given \( \mathcal{G} \), and 2) \( \beta (z, 0) = E_P [Y_1 | V = 0] = E_\mu \left[ \frac{f_\omega (\delta)}{f (\delta)} Y_1 \right] \) and \( \lim_{\delta \to 0^+} \beta (z, -\epsilon) = E_P [Y_0 | V = 0] = E_\mu \left[ \frac{f_\omega (\delta)}{f (\delta)} Y_0 \right] \). First, for all \( \Omega \times A \in \mathcal{G} \), we have

\( \int_{\Omega \times A} \beta (z, \delta) P (d (z, \delta)) = \int_A \int_{\Omega} Y \frac{f_\omega (\delta)}{f (\delta)} \mu (d \omega) \, d \delta \) by 18.20.c and 18.20.d of Billingsley (1995). This is equal to \( \int_{\Omega} \int_A Y \frac{f_\omega (\delta)}{f (\delta)} v (d \delta) \mu (d \omega) = \int_{\Omega} \int_A Y f_\omega (\delta) \, d \delta \mu (d \omega) \) because \( v \) has density \( f \) (see above). This is equal to \( \int_{\Omega} \int_A Y v_\omega (d \delta) \mu (d \omega) = \int_{\Omega \times A} Y P (d (\omega, \delta)) \), because \( v_\omega \) has density \( f_\omega \), and by 18.20.c of Billingsley (1995). Second, let \( \delta = 0 \). \( \int \frac{f_\omega (0)}{f (0)} \mu (d \omega) = \int \frac{f_\omega (0)}{f (0)} \mu (d \omega) = E_P [Y_1 | V = 0] \), by the definition of \( Y \), and the same argument above. Also, \( \int Y_1 \frac{f_\omega (0)}{f (0)} \mu (d \omega) = E_\mu \left[ \frac{f_\omega (0)}{f (0)} Y_1 \right] \). Finally, let \( \delta_n < 0 \), \( \delta_n \to 0 \). \( \frac{f_\omega (\delta_n)}{f (\delta_n)} \to \frac{f_\omega (0)}{f (0)} \), by D2. Need to show \( \lim_n \int Y_0 \frac{f_\omega (\delta_n)}{f (\delta_n)} \mu (d \omega) = \int Y_0 \frac{f_\omega (0)}{f (0)} \mu (d \omega) \). This follows from dominated convergence with \( |Y_0 \frac{f_\omega (\delta_n)}{f (\delta_n)}| \) dominated by \( |Y_0 \frac{g_\omega}{f (\delta_n)}| \) (same \( g_\omega \) as above). By the same argument as above, \( \int Y_0 \frac{f_\omega (0)}{f (0)} \mu (d \omega) = E_P [Y_0 | V = 0] = E_\mu \left[ \frac{f_\omega (0)}{f (0)} Y_0 \right] \).

to A4, for every \( x_0 \in \mathbb{R} \), \( F \equiv [\omega : X (\omega) \leq x_0] \) is in \( \mathcal{F} \), and thus c) follows from a).

**Proposition 3**

If Conditions A1-A5, B2, C2, and D2 hold, then:

a) and c) of Proposition 2 are true, and
b) \( E_P [Y|V = 0] - \lim_{\varepsilon \to 0^+} E_P [Y|V = -\varepsilon] = E_\mu \left[ \frac{f_\omega(0)}{f(0)} (Y^+ - Y^-) \right] \equiv ATE^{**} \)

**Proof.** For a) and c), see the proof to Proposition 2. b) First, following the argument the proof to Proposition 2, \( \beta(z, \delta) \) is a version of the conditional expectation of \( Y \) given \( G \). Second, let \( \delta = 0 \).

\[
\int \Omega Y \frac{f_\omega(0)}{f(0)} \mu (d\omega) = \int \Omega Y^+ \frac{f_\omega(0)}{f(0)} \mu (d\omega) = E_\mu \left[ \frac{f_\omega(0)}{f(0)} Y^+ \right].
\]

Finally, let \( \delta_n < 0, \delta_n \to 0 \). \( \frac{f_\omega(\delta_n)}{f(\delta_n)} \to \frac{f_\omega(0)}{f(0)} \), by D2. Need to show \( \lim_n \int \Omega Y \frac{f_\omega(\delta_n)}{f(\delta_n)} \mu (d\omega) = \int \Omega Y - \frac{f_\omega(0)}{f(0)} \mu (d\omega) \). This follows from dominated convergence with \( |Y \frac{f_\omega(\delta_n)}{f(\delta_n)}| \) dominated by \( |h_\omega \frac{g_\omega}{\inf y(\omega, \delta_n)}| \) (same \( g_\omega \) as above) where \( h_\omega \equiv \sup_n |y(\omega, \delta_n)| \), which is finite for each \( \omega \), because \( y(\omega, \delta_n) \to Y^- \), by C2. It follows that \( \int \Omega Y^- \frac{f_\omega(0)}{f(0)} \mu (d\omega) = E_\mu \left[ \frac{f_\omega(0)}{f(0)} Y^- \right] \).