Appendix: For Online Publication

Derivation of (2):

$$\frac{t^2}{t_{AR}^2} = \frac{\frac{(\hat{\beta}_{IV}-\beta)^2}{\hat{V}_N(\hat{\beta}_{IV})}}{\frac{\hat{\pi}^2\left(\hat{\beta}_{IV}-\beta\right)^2}{\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)}} = \frac{\frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)}{\hat{V}_N\left(\hat{\beta}_{IV}\right)}$$

Focus on the denominator to show that $\hat{V}_N\left(\hat{\beta}_{IV}\right) = \frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)\left[1-2\hat{\rho}\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}\right]$.

Let $\hat{u}^0$ denote the residual based on imposing the null. Then, $\hat{u} = \hat{u}^0 - \hat{v}(\hat{\beta}_{IV}-\beta)$.

Using **bold** to indicate matrices,

$$\begin{aligned}
\hat{V}_N\left(\hat{\beta}_{IV}\right) &= (\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{u}) \\
&= \underbrace{(\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{u}^0)}_{A} - \underbrace{2(\hat{\beta}_{IV}-\beta)(\mathbf{Z}'\mathbf{X})^{-2}\hat{C}(Z\hat{u}^0, Z\hat{v})}_{B} + \underbrace{(\hat{\beta}_{IV}-\beta)^2(\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{v})}_{C}
\end{aligned}$$

$$A: \quad (\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{u}^0) = \frac{1}{[(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^2}(\mathbf{Z}'\mathbf{Z})^{-2}\hat{V}(Z\hat{u}^0) = \frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)$$

$$\begin{aligned}
C: \quad (\hat{\beta}_{IV}-\beta)^2(\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{v}) &= (\hat{\beta}_{IV}-\beta)^2\frac{1}{[(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}]^2}(\mathbf{Z}'\mathbf{Z})^{-2}\hat{V}(Z\hat{v}) \\
&= (\hat{\beta}_{IV}-\beta)^2\frac{1}{f^2} = \frac{\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)}{\hat{\pi}^2}\frac{t_{AR}^2}{f^2}
\end{aligned}$$

Note that $sgn\left(\frac{t_{AR}}{f}\right) = sgn\left(\frac{\hat{\pi}(\hat{\beta}_{IV}-\beta)}{\hat{\pi}}\right) = sgn\left(\hat{\beta}_{IV}-\beta\right)$.

$$B: \quad 2(\hat{\beta}_{IV}-\beta)(\mathbf{Z}'\mathbf{X})^{-2}\hat{C}(Z\hat{u}^0, Z\hat{v}) = 2\hat{\rho}\frac{(\hat{\beta}_{IV}-\beta)}{\sqrt{(\hat{\beta}_{IV}-\beta)^2}}\sqrt{\underbrace{(\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{u}^0)}_{A}}\sqrt{\underbrace{(\hat{\beta}_{IV}-\beta)^2(\mathbf{Z}'\mathbf{X})^{-2}\hat{V}(Z\hat{v})}_{C}}$$

$$= 2\hat{\rho}\,sgn\left(\hat{\beta}_{IV}-\beta\right)\frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)\sqrt{\frac{t_{AR}^2}{f^2}} \; = \; \frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)\cdot 2\hat{\rho}\cdot\frac{t_{AR}}{f}$$

Putting these terms together, the formula for t^2 follows.

$$\hat{V}_N\left(\hat{\beta}_{IV}\right) \;=\; Ⓐ - Ⓑ + Ⓒ \;=\; \frac{1}{\hat{\pi}^2}\hat{V}_N\left(\hat{\pi}\left(\hat{\beta}_{IV}-\beta\right)\right)\left[1-2\hat{\rho}\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}\right]$$

Details for deriving the function $\tilde{c}(|f|)$ **such that** $\Pr\left[t^2 > \tilde{c}(|f|)\right] = 0.05$

We will consider smooth functions $\tilde{k}(|f|)$ and impose a restriction on it so that it satisfies the above condition. One can use the following (parametric) representation of any candidate function $\tilde{k}(|f|)$

$$\begin{aligned}
&\tilde{k} = k_4^*(f_0)\\
&f = r_4^*(f_0, k_4^*(f_0)) = \frac{1}{2}\left(f_0 + \sqrt{f_0}\sqrt{f_0 + 4\sqrt{k_4^*(f_0)}}\right)
\end{aligned}$$

where the second line is the largest (of up to four) root when one solves

$$k_4^*(f_0) = \frac{f^2(f-f_0)^2}{f_0^2}$$

for f. $k^*(f_0)$ by definition gives the value of t^2 at the "rightmost" intersection of the curve represented in (11) and the function $\tilde{k}(|f|)$ for each value of f_0. As a consequence r_4^* has the interpretation of the f-coordinate at this intersection. Therefore, any function $k_4^*(f_0)$ implies a $\tilde{k}(|f|)$.

There are three analogous parameterizations for the other potential points where

$\tilde{k}(|f|)$ intersects the curve in (11):

$$\begin{aligned}
&\tilde{k} = k_3^*(x_3) \\
&f = r_3^*(x_3, k_3^*(x_3)) = \frac{1}{2}\left(x_3 + \sqrt{x_3}\sqrt{x_3 - 4\sqrt{k_3^*(x_3)}}\right) \\
&\tilde{k} = k_2^*(x_2) \\
&f = r_2^*(x_2, k_2^*(x_2)) = \frac{1}{2}\left(x_2 - \sqrt{x_2}\sqrt{x_2 - 4\sqrt{k_2^*(x_2)}}\right) \\
&\tilde{k} = k_1^*(x_1) \\
&f = r_1^*(x_1, k_1^*(x_1)) = \frac{1}{2}\left(x_1 - \sqrt{x_1}\sqrt{x_1 + 4\sqrt{k_1^*(x_1)}}\right)
\end{aligned}$$

where we have used the same logic as above: k_1^*, k_2^*, k_3^* are the values of t^2 at the three other points of intersection, where r_1^* is the leftmost intersection, and r_2^* and r_3^* are points that straddle the middle, concave, section of the fourth-order polynomial graph. Depending on the value of f_0, r_2^* and r_3^* may be equal or may not exist. We are using the notation x_1, x_2, x_3 as arguments to emphasize that these are three separate parameterizations.

Since we are considering a single function $\tilde{k}(|f|)$, then the first parameterization must agree with the other three:

$$\begin{aligned}
-r_1^*\left(x_1, \tilde{k}\right) &= r_4^*\left(f_0, \tilde{k}\right) \\
r_2^*\left(x_2, \tilde{k}\right) &= r_4^*\left(f_0, \tilde{k}\right) \\
r_3^*\left(x_3, \tilde{k}\right) &= r_4^*\left(f_0, \tilde{k}\right)
\end{aligned}$$

where the negative sign in the first line follows from the fact that $\tilde{k}(|f|)$ is symmetric around $f = 0$ and the values of f of the "leftmost" and "rightmost" points of intersection are always of opposite sign.[1]

[1] Note that when $f_0 > 0$, the middle two points of intersection are also positive, and when $f_0 < 0$ the two points are negative.

Substituting for the expressions for $r_1^*, r_2^*, r_3^*, r_4^*$, and then solving for x_1, x_2, x_3 yields

$$x_1\left(f_0,\tilde{k}\right) = \frac{2f_0^{\frac{5}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}+\sqrt{\tilde{k}}f_0^{\frac{3}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}+3\sqrt{\tilde{k}}f_0^2+\tilde{k}f_0+2f_0^3}{\tilde{k}-4f_0^2} \text{ when } 4f_0^2<\tilde{k}$$
$$x_2\left(f_0,\tilde{k}\right) = \frac{-2f_0^{\frac{5}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}-\sqrt{\tilde{k}}f_0^{\frac{3}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}-3\sqrt{\tilde{k}}f_0^2-\tilde{k}f_0-2f_0^3}{\tilde{k}-4f_0^2} \text{ when } \tilde{k}<4f_0^2$$
$$x_3\left(f_0,\tilde{k}\right) = \frac{-2f_0^{\frac{5}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}-\sqrt{\tilde{k}}f_0^{\frac{3}{2}}\sqrt{4\sqrt{\tilde{k}}+f_0}-3\sqrt{\tilde{k}}f_0^2-\tilde{k}f_0-2f_0^3}{\tilde{k}-4f_0^2} \text{ when } \tilde{k}<4f_0^2.$$

These three equations tell us that for a given value f_0 and its corresponding $k_4^*(f_0), r_4^*(f_0, k_4^*(f_0))$, we can find the corresponding value $x_1, x_2,$ or x_3, which are the values of $E[f]$ for another data generating process whose curve (11) also runs through the point r_4^*, k_4^*.

In summary, any well-defined function $k_4^*(f_0)$ uniquely determines the functions $k_1^*(x_3), k_2^*(x_2), k_3^*(x_1)$ via the three parameterizations:

$$\begin{aligned}
k_3^* &= k_4^*(x) \\
x_3 &= x_3(x, k_4^*(x)) \\
k_2^* &= k_4^*(x) \\
x_2 &= x_2(x, k_4^*(x)) \\
k_1^* &= k_4^*(x) \\
x_1 &= x_1(x, k_4^*(x)).
\end{aligned} \tag{A1}$$

Thus we "trace out" the functions k_1^*, k_2^*, k_3^* by varying the value of x.

With these functions in hand we can compute the acceptance probability $\Pr\left[t^2 \le \tilde{k}(|f|)\right]$ in three cases:

Case 1: $f_0 \ge f_0^*$

$$\begin{aligned}
&\Phi(r_4^*(f_0, k_4^*(f_0)) - f_0) - \Phi(r_3^*(f_0, k_3^*(f_0)) - f_0) + \\
&\Phi(r_2^*(f_0, k_2^*(f_0)) - f_0) - \Phi(r_1^*(f_0, k_1^*(f_0)) - f_0)
\end{aligned} \tag{A2}$$

Case 2: $f_0^{\min} \leq f_0 \leq f_0^*$

$$\Phi(r_4^*(f_0, k_4^*(f_0)) - f_0) - \Phi(r_3^*(f_0, k_3^*(f_0)) - f_0) + \quad \text{(A3)}$$
$$\Phi(r_3^*(f_0, k_2^*(f_0)) - f_0) - \Phi(r_1^*(f_0, k_1^*(f_0)) - f_0)$$

Case 3: $f_0 \leq f_0^{\min}$

$$\Phi(r_4^*(f_0, k_4^*(f_0)) - f_0) - \Phi(r_1^*(f_0, k_1^*(f_0)) - f_0). \quad \text{(A4)}$$

These cases are different because the polynomial curve can intersect $\tilde{k}(|f|)$ a different number of times and in different parts. In the first case, the polynomial curve intersects on either side of the middle concave portion of the polynomial curve. In the second case, the middle two intersections occur on the right side of the concave portion of the polynomial curve, and in the third case, the polynomial intersects $\tilde{k}(|f|)$ only twice. Specifically, f_0^* is the value such that $k_2^*(f_0) = \frac{f_0^2}{16}$.[2] $f_0^{\min}$ is the value of f_0 such that $k_2^*(f_0) = k_3^*(f_0)$. These three cases are illustrated in Appendix Figure 1.

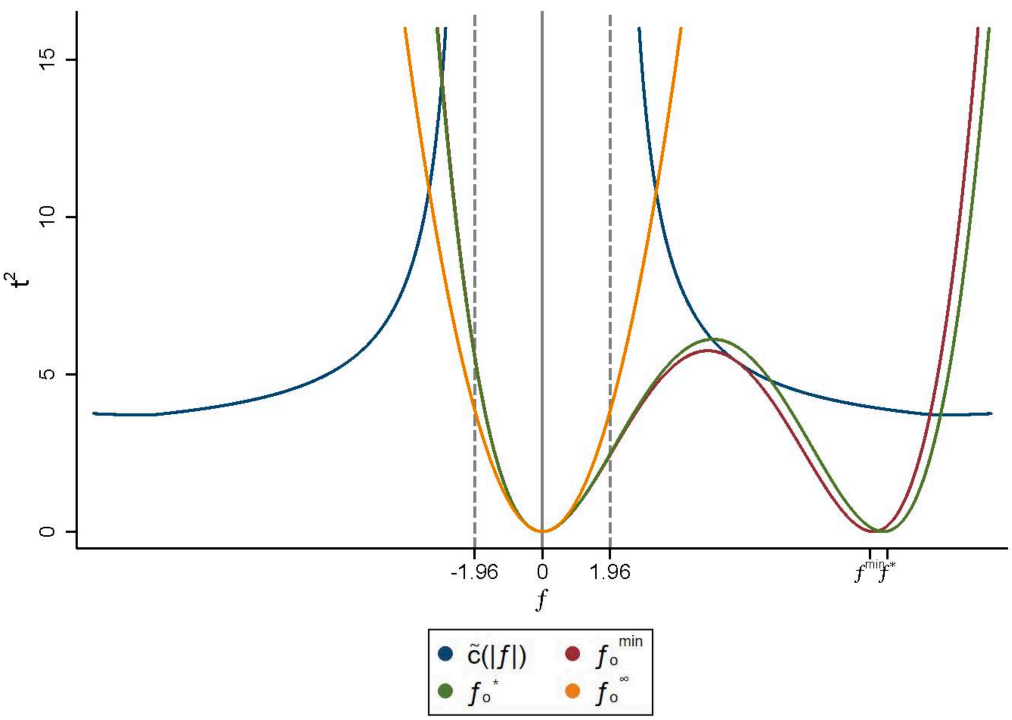
We are now in a position to find a particular function $k_4^*(x)$, call it $\tilde{c}_4^*(x)$, such that its implied $\tilde{c}(|f|)$ satisfies the desired property that $\Pr_{f_0,\rho=1}\left[t^2 > \tilde{c}(|f|)\right] = 0.05$.

This is because just as a candidate function for the solution $k_4^*(x)$ uniquely determines the functions $k_1^*(x), k_2^*(x), k_3^*(x)$ via equation (A1), the functions $k_1^*(x), k_2^*(x), k_3^*(x)$ independently determine $k_4^*(x)$ by setting the acceptance probability in (A2), (A3), (A4) to 0.95.

Therefore equations (A1), (A2), (A3), and (A4) define a functional G that maps a candidate function $k_4^*(x)$ back to the same space of functions, and the desired $c_4^*(x)$ is the fixed point that satisfies $\tilde{c}_4^* = G(\tilde{c}_4^*)$. The solution $\tilde{c}_4^*(x)$ implies a $\tilde{c}(|f|)$ and in turn the desired $c(F)$.

[2] We are trying the find the value of f_0^* such that the middle concave part of the polynomial curve intersects with $\tilde{k}(|f|)$ when the intersection point is at the local maximum of the concave portion. For f_0 values that are greater than f_0^* , then the middle two points of intersection straddle the concave portion of the curve, but for values less than f_0^*, both intersection points are on the right side. The value of f at the local maximum of the concave portion is $\frac{f_0}{2}$. Substituting in $k_2^*(f_0) = \frac{\left(\frac{f_0}{2}\right)^2\left(\frac{f_0}{2} - f_0\right)^2}{(f_0)^2} = \frac{f_0^2}{16}$.

Appendix Figure 1: $\tilde{c}\,(|f|)$ function, t^2 as a function of f, ρ=1



Derivation of Results 1a-b-c, 2a-b-c-d

Available from authors upon request.