Dual characterization of properties of risk measures on Orlicz hearts

Patrick Cheridito *
Princeton University
Princeton, NJ 08544, USA

Tianhui Li †
Cambridge University
Cambridge, CB3 0DS, UK

First version: February 26, 2008. This version: July 18, 2008

Abstract

We extend earlier representation results for monetary risk measures on Orlicz hearts. Then we give general conditions for such risk measures to be Gâteaux-differentiable, strictly monotone with respect to almost sure inequality, strictly convex modulo translation, strictly convex modulo comonotonicity, or monotone with respect to different stochastic orders. The theoretical results are used to analyze various specific examples of risk measures.

Key Words: Risk measures, Gâteaux-differentiability, strict monotonicity, strict convexity, stochastic orders, Orlicz hearts

1 Introduction

The purpose of this paper is to give characterizations of properties of risk measures that can be used to analyze particular examples. We first extend earlier representation results for risk measures on Orlicz hearts. Then we give general conditions for monetary risk measures to be Gâteaux-differentiable, strictly monotone with respect to almost sure inequality, strictly convex modulo translation, strictly convex modulo comonotonicity, or monotone with respect to different stochastic orders. The theoretical results are applied to study properties of risk measures belonging to different parametric families.

Artzner et al. (1999), Föllmer and Schied (2002a, 2002b, 2004), Frittelli and Rosazza Gianin (2002) introduced the notions of coherent, convex and monetary risk measures. The risky objects in Artzner et al. (1999) and Föllmer and Schied (2002a, 2002b, 2004) are uncertain financial positions modelled by bounded random variables. Risk measures for unbounded random variables have, among others, been studied in Frittelli and Rosazza Gianin (2002, 2004), Delbaen (2002, 2006), Cherny (2006), Rockafellar et al. (2006), Ruszczyński and Shapiro (2006), Filipović and Kupper (2006), Cheridito and Li (2007), Filipović and Svindland (2007).

Here, we work with risk measures for random variables belonging to an Orlicz heart. This allows us to build on duality results of Cheridito and Li (2007). In Section 2 we introduce the setup and extend representation results of Ruszczyński and Shapiro (2006) as well as Cheridito and Li (2007). In Section 3, we give conditions for risk measures to be differentiable in the Gâteaux-sense. Section 4 provides characterizations of risk measures that are strictly monotone with respect to almost sure inequality. In Section 5 we give conditions for risk measures to be strictly convex modulo translation

We thank Andreas Hamel and Michael Kupper for fruitful discussions and helpful comments.
*Supported by NSF Grant DMS-0642361, a Rheinstein Award and a Peek Fellowship.
†Supported by a Marshall Scholarship and a Merage Fellowship.

or comonotonicity. In Section 6 we discuss monotonicity of risk measures with respect to different stochastic orders. Section 7 studies properties of risk measures that can be obtained as cash-additive hulls of monotone convex functionals. In Section 8, we analyze five different parametric families of risk measures. Two of them were introduced in Cheridito and Li (2007), the others are new.

2 Definitions and preliminaries

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Equalities $X = Y$ and inequalities $X \geq Y$ between random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ are understood in the $\mathbb{P}$-almost sure sense. L^0 denotes the space of all random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, where two random variables are identified if they are $\mathbb{P}$-almost surely equal. For $p \in [1, \infty)$, L^p denotes the subspace of L^0 consisting of all p-integrable random variables and L^∞ the subspace of L^0 of essentially bounded random variables. Let $\Phi : [0, \infty) \to [0, \infty)$ be a convex function with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. Then Φ is automatically continuous and increasing (by which we mean that $\Phi(x) \leq \Phi(y)$ for $x \leq y$). Define the function $\Psi : [0, \infty) \to [0, \infty]$ by

$$\Psi(y) := \sup_{x \geq 0} \{xy - \Phi(x)\} .$$

The Orlicz heart

$$M^\Phi := \left\{X \in L^0 : \mathrm{E}_{\mathbb{P}} [\Phi (c|X|)] < \infty \text{ for all } c > 0\right\}$$

with the $\mathbb{P}$-almost sure ordering and the Luxemburg norm

$$\|X\|_\Phi := \inf \left\{\lambda > 0 : \mathrm{E}_{\mathbb{P}} \left[\Phi\left(\frac{|X|}{\lambda}\right)\right] \leq 1\right\}$$

is a Banach lattice, whose norm dual is the Orlicz space

$$L^\Psi := \left\{\xi \in L^0 : \mathrm{E}_{\mathbb{P}} [\Psi (c|\xi|)] < \infty \text{ for some } c > 0\right\}$$

with the Orlicz norm

$$\|\xi\|^*_\Phi := \sup \left\{\mathrm{E}_{\mathbb{P}} [X\xi] : X \in L^\Phi \, , \, \|X\|_\Phi \leq 1\right\} ,$$

which is equivalent to the Luxemburg norm $\|\cdot\|_\Psi$. For proofs of these facts, see, for instance, Edgar and Sucheston (1992).

We call a mapping $\rho : M^\Phi \to (-\infty, \infty]$ a monetary risk measure on M^Φ if it has the following properties:

(F) **Finiteness at** 0**:** $\rho(0) \in \mathbb{R}$
(M) **Monotonicity:** $\rho(X) \geq \rho(Y)$ for all $X, Y \in M^\Phi$ such that $X \leq Y$
(T) **Translation property:** $\rho(X + m) = \rho(X) - m$ for all $X \in M^\Phi$ and $m \in \mathbb{R}$

We call a monetary risk measure ρ on M^Φ convex if it also satisfies
(C) **Convexity:** $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for all $X, Y \in M^\Phi$ and $\lambda \in (0, 1)$

A convex monetary risk measure ρ on M^Φ is called coherent if it fulfills
(P) **Positive homogeneity:** $\rho(\lambda X) = \lambda\rho(X)$ for all $X \in M^\Phi$ and $\lambda \geq 0$.

It follows from (F), (M) and (T) that $\rho(L^\infty) \subset \mathbb{R}$. We identify a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ that is absolutely continuous with respect to $\mathbb{P}$ with its Radon–Nykodim derivative $\xi = d\mathbb{Q}/d\mathbb{P} \in L^1$. By $\mathcal{D}^\Psi$ we denote the set of all Radon–Nykodim densities in L^Ψ:

$$\mathcal{D}^\Psi := \left\{\xi \in L^\Psi : \xi \geq 0 \, , \, \mathrm{E}_{\mathbb{P}} [\xi] = 1\right\} ,$$

and of course,

$$\mathcal{D}^q := \{\xi \in L^q : \xi \geq 0\,,\, \mathrm{E}_{\mathbb{P}}[\xi] = 1\}\,, \quad \text{for } q \in [1,\infty]\,.$$

We call a mapping $\gamma : \mathcal{D}^\Psi \to (-\infty,\infty]$ a penalty function if it is bounded from below and not identically equal to ∞. We say a penalty function on $\mathcal{D}^\Psi$ satisfies the growth condition (G) if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that

$$\gamma(\mathbb{Q}) \geq a + b\,\|\mathbb{Q}\|_\Phi^* \quad \text{for all } \mathbb{Q} \in \mathcal{D}^\Psi\,. \tag{2.1}$$

Since $\|.\|_\Psi$ and $\|.\|_\Phi^*$ are equivalent norms on L^Ψ, (2.1) holds if and only if there exist constants $a' \in \mathbb{R}$ and $b' > 0$ such that

$$\gamma(\mathbb{Q}) \geq a' + b'\,\|\mathbb{Q}\|_\Psi \quad \text{for all } \mathbb{Q} \in \mathcal{D}^\Psi\,.$$

For a function $f : M^\Phi \to (-\infty,\infty]$, we denote

$$\operatorname{dom} f := \left\{X \in M^\Phi : f(X) < \infty\right\}\,.$$

If f is convex, then so is $\operatorname{dom} f$. Unless otherwise specified, we call f continuous, lower semicontinuous or Lipschitz-continuous if it is so with respect to $\|.\|_\Phi$. By $\operatorname{int}(A)$ we denote the interior of a subset $A \subset M^\Phi$ with respect to the norm-topology, and by $\operatorname{core}(A)$ the algebraic interior, that is, the set of all points $x \in A$ with the property that for every $y \in M^\Phi$, there exists $\varepsilon > 0$ such that $x + ty \in A$ for all $t \in [0,\varepsilon]$. $\operatorname{int}(A)$ is always contained in $\operatorname{core}(A)$, but not necessarily the other way around.

We need the following two results on the dual representation of risk measures on Orlicz hearts. They summarize and extend Theorem 2.2 of Ruszczyński and Shapiro (2006) and Theorems 4.5 and 4.6 of Cheridito and Li (2007).

Theorem 2.1. *Let γ be a penalty function on $\mathcal{D}^\Psi$. Then*

$$\rho_\gamma(X) := \sup_{\mathbb{Q}\in\mathcal{D}^\Psi} \{\mathrm{E}_\mathbb{Q}[-X] - \gamma(\mathbb{Q})\} \tag{2.2}$$

defines a lower semicontinuous convex monetary risk measure on M^Φ, and the following are equivalent:

(i) *γ satisfies the growth condition* (G)

(ii) $\operatorname{core}(\operatorname{dom}\rho_\gamma) \neq \emptyset$

(iii) *ρ_γ is real-valued and locally Lipschitz-continuous*

(iv) *For each $X \in M^\Phi$ and every sequence $(\mathbb{Q}_n)_{n\geq 1}$ in $\mathcal{D}^\Psi$ satisfying*

$$\lim_{n\to\infty} \{\mathrm{E}_{\mathbb{Q}_n}[-X] - \gamma(\mathbb{Q}_n)\} = \rho_\gamma(X)\,,$$

the sequences $\mathrm{E}_{\mathbb{Q}_n}[X]$ and $\gamma(\mathbb{Q}_n)$, $n \geq 1$, are bounded.

If (i)–(iv) *hold and γ is $\sigma(\mathcal{D}^\Psi, M^\Phi)$-lower semicontinuous, then*

$$\rho_\gamma(X) = \max_{\mathbb{Q}\in\mathcal{D}^\Psi} \{\mathrm{E}_\mathbb{Q}[-X] - \gamma(\mathbb{Q})\} \quad \textit{for all } X \in M^\Phi\,. \tag{2.3}$$

Proof. That (2.2) defines a lower semicontinuous convex monetary risk measure on M^Φ is clear. The equivalence of (i)–(iii) is shown in Theorem 4.5 of Cheridito and Li (2007). Clearly, it follows from (iv) that ρ_γ is real-valued. In particular, one has (iv) $\Rightarrow$ (ii). So the equivalence of (i)–(iv) is proved if we can show (iii) $\Rightarrow$ (iv). To do that, we assume (iii) and choose $X \in M^\Phi$ and a sequence $(\mathbb{Q}_n)_{n\geq 1}$ in $\mathcal{D}^\Psi$ such that

$$\mathrm{E}_{\mathbb{Q}_n}[-X] - \gamma(\mathbb{Q}_n) \to \rho_\gamma(X) \in \mathbb{R}\,. \tag{2.4}$$

Suppose that $(\gamma(\mathbb{Q}_n))_{n\geq 1}$ is unbounded. Since γ is bounded from below, we can, by possibly passing to a subsequence, assume that $\gamma(\mathbb{Q}_n) \geq n$, for all $n \geq 1$. Then it follows from (2.4) that $\mathrm{E}_{\mathbb{Q}_n}[-X]/\gamma(\mathbb{Q}_n) \to 1$. In particular, $\mathrm{E}_{\mathbb{Q}_n}[-X] \to \infty$, and therefore,

$$\rho_\gamma(2X) \geq \mathrm{E}_{\mathbb{Q}_n}[-2X] - \gamma(\mathbb{Q}_n) = \mathrm{E}_{\mathbb{Q}_n}[-X] + \mathrm{E}_{\mathbb{Q}_n}[-X] - \gamma(\mathbb{Q}_n) \to \infty\,,$$

a contradiction to (iii). Hence, $(\gamma(\mathbb{Q}_n))_{n\geq 1}$ must be bounded, which, by (2.4), implies that also $(\mathrm{E}_{\mathbb{Q}_n}[X])_{n\geq 1}$ is bounded.

It remains to show (2.3) if γ satisfies (i)–(iv) and is $\sigma(\mathcal{D}^\Psi, M^\Phi)$-lower semicontinuous. So assume these conditions hold and let $X \in M^\Phi$. Choose a sequence $(\mathbb{Q}_n)_{n\geq 1}$ in $\mathcal{D}^\Psi$ such that

$$\mathrm{E}_{\mathbb{Q}_n}[-X] - \gamma(\mathbb{Q}_n) \to \rho_\gamma(X)\,.$$

By (iv), $(\gamma(\mathbb{Q}_n))_{n\geq 1}$ is bounded, which together with (i) implies that $(\|\mathbb{Q}_n\|^*_\Phi)_{n\geq 1}$ is bounded. So it follows from the Alaoglu–Bourbaki theorem that there exists a subsequence of $(\mathbb{Q}_n)_{n\geq 1}$ converging to some $\mathbb{Q} \in \mathcal{D}^\Psi$ in $\sigma(\mathcal{D}^\Psi, M^\Phi)$. Since γ is $\sigma(\mathcal{D}^\Psi, M^\Phi)$-lower semicontinuous, one obtains

$$\mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q}) \geq \lim_{n\to\infty} \mathrm{E}_{\mathbb{Q}_n}[-X] - \gamma(\mathbb{Q}_n) = \rho_\gamma(X) \geq \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\,,$$

and (2.3) is proved. □

For a convex monetary risk measure $\rho : M^\Phi \to (-\infty, \infty]$, we define the function $\rho^\# : \mathcal{D}^\Psi \to (-\infty, \infty]$ by

$$\rho^\#(\mathbb{Q}) := \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{Q}}[-X] - \rho(X)\right\}\,.$$

It is obviously $\sigma(\mathcal{D}^\Psi, M^\Phi)$-lower semicontinuous. Moreover, one has:

Theorem 2.2. *For a convex monetary risk measure $\rho : M^\Phi \to (-\infty, \infty]$, the following hold:*
(i) *If ρ is lower semicontinuous, then $\rho^\#$ is a penalty function on $\mathcal{D}^\Psi$ with $\rho = \rho_{\rho^\#}$.*
(ii) *If $\rho = \rho_\gamma$ for a penalty function γ on $\mathcal{D}^\Psi$, then $\rho^\#$ is the largest convex, $\sigma(\mathcal{D}^\Psi, M^\Phi)$-lower semicontinuous minorant of γ.*
(iii) *If* $\mathrm{core}(\mathrm{dom}\,\rho) \neq \emptyset$*, then ρ is real-valued as well as locally Lipschitz-continuous, and*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{D}^\Psi} \left\{\mathrm{E}_{\mathbb{Q}}[-X] - \rho^\#(\mathbb{Q})\right\} \quad \text{for all } X \in M^\Phi\,.$$

(iv) *If ρ is coherent and* $\mathrm{core}(\mathrm{dom}\,\rho) \neq \emptyset$*, then ρ is real-valued as well as Lipschitz-continuous, and*

$$\rho(X) = \max_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[-X] \quad \text{for all } X \in M^\Phi\,,$$

where $\mathcal{Q} = \left\{\mathbb{Q} \in \mathcal{D}^\Psi : \mathrm{E}_{\mathbb{Q}}[X] + \rho(X) \geq 0 \text{ for all } X \in M^\Phi\right\}$.

Proof. (i): It follows from Theorem 2.2 in Ruszczyński and Shapiro (2006) that $\rho = \rho_{\rho^\#}$, which can only hold if $\rho^\#$ is a penalty function on $\mathcal{D}^\Psi$.

(ii) follows from Theorem 2.3.4 in Zălinescu (2002) like the last part of Theorem 4.6 in Cheridito and Li (2007).

(iii) follows from Theorems 4.5 and 4.6 of Cheridito and Li (2007).

(iv) follows from Corollaries 4.7 and 4.8 of Cheridito and Li (2007). □

3 Subdifferentiability and Gâteaux-differentiability

Let $f : M^\Phi \to (-\infty, \infty]$ be a convex function, and denote by f^* the convex conjugate given by

$$f^*(\xi) := \sup_{X\in M^\Phi} \left\{\mathrm{E}_{\mathbb{P}}[X\xi] - f(X)\right\}\,, \quad \xi \in L^\Psi\,. \tag{3.1}$$

It is immediate from (3.1) that for fixed $X \in \operatorname{dom} f$,

$$f(X) + f^*(\xi) \geq \mathrm{E}_{\mathbb{P}}[X\xi] \quad \text{for all } \xi \in L^\Psi ,$$

with equality if and only ξ is in the subdifferential

$$\partial f(X) := \left\{ \xi \in L^\Psi : f(X+Y) - f(X) \geq \mathrm{E}_{\mathbb{P}}[Y\xi] \text{ for all } Y \in M^\Phi \right\} .$$

By convexity of f, the directional derivative

$$f'(X;Y) := \lim_{\varepsilon \downarrow 0} \frac{f(X+\varepsilon Y) - f(X)}{\varepsilon} \in [-\infty, \infty]$$

exists in every direction $Y \in M^\Phi$. If there exists $\xi \in L^\Psi$ such that

$$f'(X;Y) = \mathrm{E}_{\mathbb{P}}[Y\xi] \quad \text{for all } Y \in M^\Phi ,$$

we say f is Gâteaux-differentiable at X with Gâteaux-derivative $\nabla f(X) = \xi$. If it exists, $\nabla f(X)$ is unique and

$$\mathrm{E}_{\mathbb{P}}[Y \nabla f(X)] = f'(X;Y) \geq \sup_{\xi \in \partial f(X)} \mathrm{E}_{\mathbb{P}}[Y\xi] \quad \text{for all } Y \in M^\Phi ,$$

which implies $\partial f(X) = \{\nabla f(X)\}$. On the other hand, if f is continuous at X and $\partial f(X) = \{\xi\}$ for some $\xi \in L^\Psi$, then it follows from Theorem 2.4.10 of Zălinescu (2002) that f is Gâteaux-differentiable at X with Gâteaux-derivative $\nabla f(X) = \xi$.

If ρ is a convex monetary risk measure on M^Φ, it follows from the properties (M) and (T) that

$$\rho^*(\xi) = \begin{cases} \rho^{\#}(-\xi) & \text{for } -\xi \in \mathcal{D}^\Psi \\ \infty & \text{for } -\xi \in L^\Psi \setminus \mathcal{D}^\Psi \end{cases} .$$

The following notation will be convenient:

Definition 3.1. *For a convex monetary risk measure ρ on M^Φ and a penalty function γ on $\mathcal{D}^\Psi$, we denote*

$$\begin{aligned} \chi_\rho(X) &:= \left\{ \mathbb{Q} \in \mathcal{D}^\Psi : \rho(X) + \rho^{\#}(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-X] \right\} , \quad X \in M^\Phi \\ \chi_{\rho,\gamma}(X) &:= \left\{ \mathbb{Q} \in \mathcal{D}^\Psi : \rho(X) + \gamma(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-X] \right\} , \quad X \in M^\Phi \\ \chi_\gamma(\mathbb{Q}) &:= \left\{ X \in M^\Phi : \rho_\gamma(X) + \gamma(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-X] \right\} , \quad \mathbb{Q} \in \mathcal{D}^\Psi \\ M^\Phi_\gamma &:= \left\{ X \in M^\Phi : \rho_\gamma(X) + \gamma(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-X] \text{ for some } \mathbb{Q} \in \mathcal{D}^\Psi \right\} \end{aligned}$$

Note that if $\gamma \geq \gamma'$ are two penalty functions on $\mathcal{D}^\Psi$ which induce the same convex monetary risk measure ρ on M^Φ, then $\chi_{\rho,\gamma}(X) \subset \chi_{\rho,\gamma'}(X)$ for all $X \in M^\Phi$ and $\chi_\gamma(\mathbb{Q}) \subset \chi_{\gamma'}(\mathbb{Q})$ for all $\mathbb{Q} \in \mathcal{D}^\Psi$. In particular, $\chi_{\rho,\gamma}(X) \subset \chi_\rho(X)$ and $\chi_\gamma(\mathbb{Q}) \subset \chi_{\rho^{\#}}(\mathbb{Q})$ since $\rho^{\#}$ is the minimal penalty function of ρ.

In terms of the notions introduced in Definition 3.1, Gâteaux-differentiability of convex monetary risk measures can be characterized as follows:

Proposition 3.2. *For a convex monetary risk measure ρ on M^Φ and $X \in \operatorname{dom} \rho$, the following hold:*
(i) $\chi_\rho(X) = -\partial \rho(X)$
(ii) *If ρ is Gâteaux-differentiable at X, then ρ is real-valued as well as locally Lipschitz-continuous, and $\chi_\rho(X) = \{-\nabla \rho(X)\}$*
(iii) *If $X \in \operatorname{core}(\operatorname{dom} \rho)$ and $\chi_\rho(X) = \{\xi\}$ for some $\xi \in \mathcal{D}^\Psi$, then ρ is Gâteaux-differentiable at X with $\nabla \rho(X) = -\xi$.*

Proof. (i) is immediate from the definition of χ_ρ and the fact that $\partial\rho(X) \subset -\mathcal{D}^\Psi$.

To prove (ii), note that X has to be in core(dom ρ) if ρ is Gâteaux-differentiable at X. Then it follows from Theorem 2.2 that ρ is real-valued and locally Lipschitz-continuous. $\chi_\rho(X) = \{-\nabla\rho(X)\}$ follows from $\partial\rho(X) = \{\nabla\rho(X)\}$ and (i).

If the assumptions of (iii) hold, it follows from (i) that $\partial\rho(X) = \{-\xi\}$. By Theorem 2.2, ρ is real-valued and continuous on M^Φ. So Theorem 2.4.10 of Zălinescu (2002) yields that ρ is Gâteaux-differentiable at X with $\nabla\rho(X) = -\xi$. □

Remark 3.3. Let ρ be a coherent risk measure on M^Φ of the form

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}} \mathrm{E}_{\mathbb{Q}}[-X]$$

for a set $\mathcal{Q} \subset \mathcal{D}^\Psi$. Then $\rho^\#(\mathbb{Q}) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$, and therefore, $\mathcal{Q} \subset \chi_\rho(m)$ for all $m \in \mathbb{R}$. So by (ii) of Proposition 3.2, ρ can only be Gâteaux-differentiable at m if $\mathcal{Q}$ consist of just one element.

4 Strict monotonicity

Definition 4.1. *We call a risk measure ρ on M^Φ strictly monotone on a subset A of M^Φ if $\rho(X) > \rho(Y)$ for all $X, Y \in A$ such that $X \le Y$ as well as $\mathbb{P}[X < Y] > 0$, and we denote $\mathcal{D}^\Psi_{\mathrm{s}} := \left\{\xi \in \mathcal{D}^\Psi : \xi > 0\right\}$.*

Theorem 4.2. *Let γ be a penalty function on $\mathcal{D}^\Psi$. Then the following are equivalent:*

(i) *ρ_γ is strictly monotone on M^Φ_γ*

(ii) *$\rho_\gamma(X) = \max_{\mathbb{Q}\in\mathcal{D}^\Psi_{\mathrm{s}}} \{\mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\}$ for all $X \in M^\Phi_\gamma$*

(iii) *$\chi_{\rho_\gamma,\gamma}(X) \subset \mathcal{D}^\Psi_{\mathrm{s}}$ for all $X \in M^\Phi_\gamma$*

(iv) *$\chi_\gamma(\mathbb{Q}) = \emptyset$ for all $\mathbb{Q} \in \mathcal{D}^\Psi \setminus \mathcal{D}^\Psi_{\mathrm{s}}$*

Proof. The implications (iv) $\Leftrightarrow$ (iii) $\Rightarrow$ (ii) are clear. So it suffices to show (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i): Let $X, Y \in M^\Phi_\gamma$ such that $X \le Y$ and $\mathbb{P}[X < Y] > 0$. Then there exists $\mathbb{Q} \in \mathcal{D}^\Psi_{\mathrm{s}}$ such that

$$\rho_\gamma(Y) = \mathrm{E}_{\mathbb{Q}}[-Y] - \gamma(\mathbb{Q}) < \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q}) \le \rho_\gamma(X)\,.$$

(i) $\Rightarrow$ (iii): Assume there exist $X \in M^\Phi_\gamma$ and $\mathbb{Q} \in \mathcal{D}^\Psi \setminus \mathcal{D}^\Psi_{\mathrm{s}}$ such that $\rho_\gamma(X) = \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})$. Then there exists $A \in \mathcal{F}$ with $\mathbb{P}[A] > 0$ and $\mathbb{Q}[A] = 0$. So

$$\rho_\gamma(X + 1_A) \ge \mathrm{E}_{\mathbb{Q}}[-X - 1_A] - \gamma(\mathbb{Q}) = \mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q}) = \rho_\gamma(X) \ge \rho_\gamma(X + 1_A)\,.$$

This implies $X + 1_A \in M^\Phi_\gamma$ and hence contradicts (i). □

Remark 4.3. If a risk measure is strictly monotone on M^Φ, it is of course also relevant (see Definition 3.4 in Delbaen (2002) for the coherent case and Definition 4.32 in Föllmer and Schied (2004) for the convex monetary case). Dual conditions for relevance of coherent risk measures are given in Theorem 3.5 of Delbaen (2002). For the convex monetary case, see Lemma 3.22 and Theorem 3.23 in Cheridito et al. (2006).

5 Strict convexity

In order to define the notion of strict convexity for monetary risk measures, we must first dispense with a trivial case. Call $X, Y \in M^\Phi$ translationally equivalent (denoted $X \sim_{\mathrm{t}} Y$) if there exists $m \in \mathbb{R}$ such that $X = Y + m$. If ρ is a monetary risk measure on M^Φ and X, Y are elements of M^Φ such that $X = Y + m$ for $m \in \mathbb{R}$, then

$$\rho(\lambda X + (1-\lambda)Y) = \rho(\lambda(Y+m) + (1-\lambda)Y) = \rho(Y) - \lambda m = \lambda\rho(X) + (1-\lambda)\rho(Y) \tag{5.1}$$

for all $0 \le \lambda \le 1$. So ρ cannot be strictly convex between X and Y.

Definition 5.1. *We call a monetary risk measure ρ on M^Φ strictly convex modulo translation on a subset A of M^Φ if*

$$\rho(\lambda X + (1-\lambda)Y) < \lambda\rho(X) + (1-\lambda)\rho(Y)$$

for all $X, Y \in A$ and $\lambda \in (0,1)$ such that $X \not\sim_{\mathrm{t}} Y$ and $\lambda X + (1-\lambda)Y \in A$.

It can be seen from (5.1) that a monetary risk measure ρ on M^Φ which is strictly convex modulo translation on a subset A of M^Φ is also convex on A.

Proposition 5.2. *A monetary risk measure on M^Φ which is strictly convex modulo translation on a convex subset A of M^Φ is also strictly monotone on A.*

Proof. Assume ρ is a monetary risk measure on M^Φ that is strictly convex modulo translation but not strictly monotone on a convex set $A \subset M^\Phi$. Then there exist $X, Y \in A$ such that $X \le Y$, $\mathbb{P}[X < Y] > 0$ and $\rho(X) = \rho(Y)$. X and Y cannot be translationally equivalent. But

$$\rho(Y) \le \rho\left(\frac{X+Y}{2}\right) \le \rho(X) = \rho(Y)$$

and therefore,

$$\rho\left(\frac{X+Y}{2}\right) = \frac{\rho(X) + \rho(Y)}{2},$$

a contradiction. □

The following theorem gives dual conditions for strict convexity modulo translation. Note that for a penalty function $\gamma : \mathcal{D}^\Psi \to (-\infty, \infty]$ and $\mathbb{Q} \in \mathcal{D}^\Psi$, a random variable $X \in M^\Phi$ is in $\chi_\gamma(\mathbb{Q})$ if and only if $X + m$ is in $\chi_\gamma(\mathbb{Q})$ for all $m \in \mathbb{R}$.

Theorem 5.3. *For a penalty function γ on $\mathcal{D}^\Psi$, the following are equivalent:*
(i) *ρ_γ is strictly convex modulo translation on M^Φ_γ*
(ii) *$\chi_{\rho_\gamma,\gamma}(X) \setminus \chi_{\rho_\gamma,\gamma}(Y) \neq \emptyset$ for all $X, Y \in M^\Phi_\gamma$ such that $X \not\sim_{\mathrm{t}} Y$*
(iii) *$\chi_{\rho_\gamma,\gamma}(X) \cap \chi_{\rho_\gamma,\gamma}(Y) = \emptyset$ for all $X, Y \in M^\Phi_\gamma$ such that $X \not\sim_{\mathrm{t}} Y$*
(iv) *for all $\mathbb{Q} \in \mathcal{D}^\Psi$, $\chi_\gamma(\mathbb{Q})$ contains at most one element modulo translation.*

Proof. The equivalence of (iii) and (iv) is obvious. So it is enough to show (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii) holds since for $X \in M^\Phi_\gamma$, $\chi_{\rho_\gamma,\gamma}(X)$ is not empty.

(ii) $\Rightarrow$ (i): Let $X, Y \in M^\Phi_\gamma$ and $\lambda \in (0,1)$ such that $X \not\sim_{\mathrm{t}} Y$ and $\lambda X + (1-\lambda)Y \in M^\Phi_\gamma$. Then $\lambda X + (1-\lambda)Y \not\sim_{\mathrm{t}} X$. Thus there exists $\mathbb{Q} \in \chi_{\rho_\gamma,\gamma}(\lambda X + (1-\lambda)Y) \setminus \chi_{\rho_\gamma,\gamma}(X)$, and we have

$$\begin{aligned} & \rho_\gamma(\lambda X + (1-\lambda)Y) = \mathrm{E}_{\mathbb{Q}}[-\lambda X - (1-\lambda)Y] - \gamma(\mathbb{Q}) \\ = \ & \lambda\left(\mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\right) + (1-\lambda)\left(\mathrm{E}_{\mathbb{Q}}[-Y] - \gamma(\mathbb{Q})\right) < \lambda\rho_\gamma(X) + (1-\lambda)\rho_\gamma(Y). \end{aligned}$$

(i) $\Rightarrow$ (iii): Assume there exist $X, Y \in M^\Phi_\gamma$ and $\mathbb{Q} \in \mathcal{D}^\Psi$ such that $X \not\sim_{\mathrm{t}} Y$ and $\mathbb{Q} \in \chi_{\rho_\gamma,\gamma}(X) \cap \chi_{\rho_\gamma,\gamma}(Y)$. Then

$$\rho_\gamma\left(\frac{X+Y}{2}\right) \ge \mathrm{E}_{\mathbb{Q}}\left[-\frac{X+Y}{2}\right] - \gamma(\mathbb{Q}) = \frac{1}{2}\left(\rho_\gamma(X) + \rho_\gamma(Y)\right) \ge \rho_\gamma\left(\frac{X+Y}{2}\right). \tag{5.2}$$

So $(X+Y)/2 \in M^\Phi_\gamma$, and (5.2) is in contradiction to (i). This shows that (i) implies (iii). □

If ρ is a coherent risk measure on M^Φ, it is linear on all rays $\{\lambda X : \lambda \geq 0\}$, $X \in M^\Phi$. So it cannot be strictly convex modulo translation. But it can be strictly convex modulo weaker equivalence relations, such as, for instance, comonotonicity. We call two random variables X and Y comonotone and write $X \sim_c Y$ if $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$ for $\mathbb{P} \times \mathbb{P}$-almost all (ω, ω'), and we define strict convexity modulo comonotonicity analogously to strict convexity modulo translation (see Definition 5.1 above).

Theorem 5.4. *For a penalty function γ on $\mathcal{D}^\Psi$, the following are equivalent:*
(i) *ρ_γ is strictly convex modulo comonotonicity on M^Φ_γ*
(ii) *$\chi_{\rho_\gamma,\gamma}(X) \setminus \chi_{\rho_\gamma,\gamma}(Y) \neq \emptyset$ for all $X, Y \in M^\Phi_\gamma$ such that $X \not\sim_c Y$*
(iii) *$\chi_{\rho_\gamma,\gamma}(X) \cap \chi_{\rho_\gamma,\gamma}(Y) = \emptyset$ for all $X, Y \in M^\Phi_\gamma$ such that $X \not\sim_c Y$*
(iv) *for all $\mathbb{Q} \in \mathcal{D}^\Psi$, $\chi_\gamma(\mathbb{Q})$ contains at most one element modulo comonotonicity.*

Proof. The implications (iii) $\Leftrightarrow$ (iv), (iii) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow exactly as in Theorem 5.3. To prove (ii) $\Rightarrow$ (i), suppose there exist $X, Y \in M^\Phi_\gamma$ and $\lambda \in (0,1)$ such that $X \not\sim_c Y$ and $\lambda X + (1-\lambda)Y \in M^\Phi_\gamma$. Assume that

$$\chi_{\rho_\gamma,\gamma}(\lambda X + (1-\lambda)Y) \subset \chi_{\rho_\gamma,\gamma}(X) \cap \chi_{\rho_\gamma,\gamma}(Y)\,. \tag{5.3}$$

Then there exists

$$\mathbb{Q} \in \bigcap_{\mu \in [0,1]} \chi_{\rho_\gamma,\gamma}(\mu X + (1-\mu)Y)\,, \tag{5.4}$$

which implies

$$\rho_\gamma(\mu X + (1-\mu)Y) = \mu \rho_\gamma(X) + (1-\mu)\rho_\gamma(Y) \quad \text{for all } \mu \in [0,1]\,. \tag{5.5}$$

But for $\mu_0 \in (0,1)$ small enough, one has $\mu_0 X + (1-\mu_0)Y \not\sim_c X$. By (5.4), $\mu_0 X + (1-\mu_0)Y$ belongs to M^Φ_γ. Hence, it follows from (ii) that there exists $\mathbb{Q}_0 \in \chi_{\rho_\gamma,\gamma}(\mu_0 X + (1-\mu_0)Y) \setminus \chi_{\rho_\gamma,\gamma}(X)$, and we obtain

$$\begin{aligned}
& \rho_\gamma(\mu_0 X + (1-\mu_0)Y) = \mathrm{E}_{\mathbb{Q}_0}\left[-\mu_0 X - (1-\mu_0)Y\right] - \gamma(\mathbb{Q}_0) \\
= \;& \mu_0 \left(\mathrm{E}_{\mathbb{Q}_0}[-X] - \gamma(\mathbb{Q}_0)\right) + (1-\mu_0)\left(\mathrm{E}_{\mathbb{Q}_0}[-Y] - \gamma(\mathbb{Q}_0)\right) < \mu_0 \rho_\gamma(X) + (1-\mu_0)\rho_\gamma(Y)\,,
\end{aligned}$$

a contradiction to (5.5). So (5.3) cannot hold, that is, there exists $\mathbb{Q} \in \chi_{\rho_\gamma,\gamma}(\lambda X + (1-\lambda)Y)$ which does not belong to $\chi_{\rho_\gamma,\gamma}(X) \cap \chi_{\rho_\gamma,\gamma}(Y)$, and we obtain

$$\begin{aligned}
& \rho_\gamma(\lambda X + (1-\lambda)Y) = \mathrm{E}_{\mathbb{Q}}\left[-\lambda X - (1-\lambda)Y\right] - \gamma(\mathbb{Q}) \\
= \;& \lambda \left(\mathrm{E}_{\mathbb{Q}}[-X] - \gamma(\mathbb{Q})\right) + (1-\lambda)\left(\mathrm{E}_{\mathbb{Q}}[-Y] - \gamma(\mathbb{Q})\right) < \lambda \rho_\gamma(X) + (1-\lambda)\rho_\gamma(Y)\,.
\end{aligned}$$

□

6 Risk measures and stochastic orders

For a random variable $X \in L^0$ with distribution function F^X we denote by q^X the right-continuous quantile function from $(0,1)$ to $\mathbb{R}$ given by

$$q^X(y) := \inf\left\{x \in \mathbb{R} : F^X(x) > y\right\}\,.$$

Viewed as a random variable on $(0,1)$ equipped with the Borel sigma-algebra and the Lebesgue measure, q^X has the same distribution as X; in particular, $\int_0^1 q^X(y)dy = \mathrm{E}_{\mathbb{P}}[X]$ for $X \in L^1$.

We call a function $f : \mathbb{R} \to \mathbb{R}$ increasing (decreasing) if $f(x) \leq f(y)$ ($f(x) \geq f(y)$) for $x \leq y$.

Definition 6.1. *Let $X, Y \in L^1$ and $\mathcal{S}$ a class of functions $f : \mathbb{R} \to \mathbb{R}$. Then we say X dominates Y with respect to $\mathcal{S}$ and write $X \succeq_{\mathcal{S}} Y$ if*

$$\mathrm{E}_{\mathbb{P}}[f(X)] \geq \mathrm{E}_{\mathbb{P}}[f(Y)]$$

for all $f \in \mathcal{S}$ such that $f(X), f(Y) \in L^1$. If $X \succeq_{\mathcal{S}} Y$ and $X \preceq_{\mathcal{S}} Y$, we call X equivalent to Y with respect to $\mathcal{S}$ and write $X \sim_{\mathcal{S}} Y$. If $X \succeq_{\mathcal{S}} Y$ and $X \not\sim_{\mathcal{S}} Y$, we say X strictly dominates Y with respect to $\mathcal{S}$ and write $X \succ_{\mathcal{S}} Y$. By i *we denote the class of all increasing functions, by* cv *all concave functions, by* icv *all increasing concave functions, and by* icx *all increasing convex functions.*

It is immediate from Definition 6.1 that $\succeq_{\mathrm{i}}$ and $\succeq_{\mathrm{cv}}$ are stronger than $\succeq_{\mathrm{icv}}$ and that $X \succeq_{\mathrm{icv}} Y$ is equivalent to $-X \preceq_{\mathrm{icx}} -Y$. Moreover, one has the following:

$$X \succeq_{\mathrm{i}} Y \;\Leftrightarrow\; q^X(y) \geq q^Y(y) \;\text{ for all } y \in (0,1) \tag{6.1}$$

$$\begin{aligned} &X \succeq_{\mathrm{icv}} Y \;\Leftrightarrow\; \textstyle\int_0^z q^X(y)dy \geq \int_0^z q^Y(y)dy \;\text{ for all } z \in (0,1) \;\Leftrightarrow\; \text{there exists a probability} \\ &\text{space with random variables } \tilde{X} \text{ and } \tilde{Y} \text{ such that } F^{\tilde{X}} = F^X,\, F^{\tilde{Y}} = F^Y \text{ and } \tilde{X} \geq \mathrm{E}[\tilde{Y} \mid \tilde{X}] \end{aligned} \tag{6.2}$$

$$\begin{aligned} &X \succeq_{\mathrm{cv}} Y \;\Leftrightarrow\; X \succeq_{\mathrm{icv}} Y \text{ and } \mathrm{E}_{\mathbb{P}}[X] = \mathrm{E}_{\mathbb{P}}[Y] \;\Leftrightarrow\; \text{there exists a probability space with} \\ &\text{random variables } \tilde{X} \text{ and } \tilde{Y} \text{ such that } F^{\tilde{X}} = F^X,\, F^{\tilde{Y}} = F^Y \text{ and } \tilde{X} = \mathrm{E}[\tilde{Y} \mid \tilde{X}]\,. \end{aligned} \tag{6.3}$$

Proofs of these facts and more on stochastic orders can, for instance, be found in Müller and Stoyan (2002), Föllmer and Schied (2004) or Shaked and Shanthikumar (2007). It is clear from (6.1)–(6.3) that

$$X \sim_{\mathrm{i}} Y \;\Leftrightarrow\; X \sim_{\mathrm{cv}} Y \;\Leftrightarrow\; X \sim_{\mathrm{icv}} Y \;\Leftrightarrow\; q^X = q^Y \;\Leftrightarrow\; F^X = F^Y\,, \tag{6.4}$$

and one obtains from Jensen's inequality for conditional expectations that $\mathrm{E}_{\mathbb{P}}[X \mid \mathcal{G}] \succeq_{\mathrm{cv}} X$ for all $X \in L^1$ and every sub-sigma-algebra $\mathcal{G} \subset \mathcal{F}$. Furthermore, if $X \in L^1$ is not $\mathcal{G}$-measurable, then $\mathrm{E}_{\mathbb{P}}[f(\mathrm{E}_{\mathbb{P}}[X \mid \mathcal{G}])] > \mathrm{E}_{\mathbb{P}}[f(X)]$ for all strictly concave functions f such that $f(X) \in L^1$. In particular, $\mathrm{E}_{\mathbb{P}}[X \mid \mathcal{G}] \succ_{\mathrm{cv}} X$ as well as $\mathrm{E}_{\mathbb{P}}[X \mid \mathcal{G}] \succ_{\mathrm{icv}} X$. On the other hand, if $X \succ_{\mathrm{cv}} Y$ or $X \succ_{\mathrm{icv}} Y$, then by (6.2)–(6.3), there exists a probability space with random variables $\tilde{X}$ and $\tilde{Y}$ distributed as X and Y, respectively, such that $\tilde{X} \geq \mathrm{E}[\tilde{Y} \mid \tilde{X}]$. Since $F^{\tilde{X}} \neq F^{\tilde{Y}}$, one has $\mathbb{P}[\tilde{X} > \mathrm{E}[\tilde{Y} \mid \tilde{X}]] > 0$ or $\mathrm{E}[\tilde{Y} \mid \tilde{X}] \neq \tilde{Y}$. This shows that $\mathrm{E}_{\mathbb{P}}[f(X)] > \mathrm{E}_{\mathbb{P}}[f(Y)]$ for all strictly concave increasing functions f such that $f(X), f(Y) \in L^1$.

Definition 6.2. *Let ρ be a monetary risk measure on M^Φ and $\mathcal{S}$ a class of functions $f : \mathbb{R} \to \mathbb{R}$. Then we call ρ $\mathcal{S}$-monotone if $\rho(X) \geq \rho(Y)$ for all $X, Y \in M^\Phi$ such that $X \preceq_{\mathcal{S}} Y$. If ρ is $\mathcal{S}$-monotone and $\rho(X) > \rho(Y)$ for all $X, Y \in M^\Phi$ such that $X \prec_{\mathcal{S}} Y$, we call ρ strictly $\mathcal{S}$-monotone. If $\rho(X)$ only depends on F^X, we call ρ distribution-based.*

Since $\succeq_{\mathrm{icv}}$ is weaker than $\succeq_{\mathrm{i}}$ and $\succeq_{\mathrm{cv}}$, an icv-monotone monetary risk measure is also i- and cv-monotone. On the other hand, the extension of Proposition 2.1 of Dana (2005) to Orlicz hearts yields that every cv-monotone monetary risk measure on M^Φ is icv-monotone. However, an i-monotone monetary risk measure is not necessarily cv- or icv-monotone:

Example 6.3. By (6.1), the monetary risk measure value-at-risk $\mathrm{VaR}_\alpha(X) := -q^X(\alpha)$ is i-monotone and distribution-based but not strictly i-monotone. Also, for every non-constant random variable $X \in L^1$, one has $X \prec_{\mathrm{cv}} Y := \mathrm{E}_{\mathbb{P}}[X]$. But there exists $\alpha \in (0,1)$ such that $\mathrm{VaR}_\alpha(X) < \mathrm{VaR}_\alpha(Y)$. So VaR_α is not cv-monotone and therefore also not icv-monotone. It is also not convex and hence not coherent (see Artzner et al., 1999; or Föllmer and Schied, 2004).

In view of (6.4), every i-, cv- or icv-monotone monetary risk measure is also distribution-based. On the other hand, Theorem 4.1 of Dana (2005) (extended to Orlicz hearts) shows that if the probability space is atomless, then every lower semicontinuous distribution-based convex monetary risk measure is icv-monotone. The following example shows that this does not need to be the case if the probability space has atoms:

Example 6.4. Consider a probability space Ω consisting of two elements, ω_1 and ω_2, and two probability measures $\mathbb{P}$ and $\mathbb{Q}$ such that

$$\mathbb{P}[\omega_1] = \mathbb{Q}[\omega_2] = \frac{1}{3} \quad \text{and} \quad \mathbb{P}[\omega_2] = \mathbb{Q}[\omega_1] = \frac{2}{3}\,.$$

Then $\rho(X) := \mathrm{E}_{\mathbb{Q}}[-X]$ defines a continuous strictly monotone coherent risk measure on L^1, which since $\mathbb{P}$ assigns weights unevenly, is distribution-based. Now consider the random variables X, Y, Z given by $X(\omega_j) = -j$, $Y(\omega_j) = j - 3$ and $Z = \mathrm{E}_{\mathbb{P}}[X] = -5/3$. Then $X \prec_{\mathrm{i}} Y$ and $X \prec_{\mathrm{cv}} Z$ but $\rho(X) = 4/3 < \rho(Y) = \rho(Z) = 5/3$. So ρ is not i-monotone, not cv-monotone, and therefore also not icv-monotone.

6.1 icv-monotone monetary risk measures and sets of acceptable positions

It is well-known that a monetary risk measure $\rho : M^\Phi \to (-\infty, \infty]$ can be reconstructed from its acceptance set

$$\mathcal{C} := \left\{X \in M^\Phi : \rho(X) \le 0\right\}$$

through

$$\rho_{\mathcal{C}}(X) := \inf\left\{m \in \mathbb{R} : X + m \in \mathcal{C}\right\}\,, \quad X \in M^\Phi\,.$$

Moreover, if $\mathcal{B}$ is a subset of M^Φ with the following three properties:

$$\text{for all } X \in \mathcal{B}, \text{ the set } \left\{Y \in M^\Phi : Y \ge X\right\} \text{ is contained in } \mathcal{B} \tag{6.5}$$

$$\rho_{\mathcal{B}}(0) \in \mathbb{R} \tag{6.6}$$

$$\rho_{\mathcal{B}}(X) \in (-\infty, \infty] \text{ for all } X \in M^\Phi\,, \tag{6.7}$$

then $\rho_{\mathcal{B}}$ is a monetary risk measure on M^Φ. In regard to the icv-order, one has the following

Proposition 6.5. *The acceptance set $\mathcal{C}$ of an* icv*-monotone monetary risk measure ρ on M^Φ has the following property:*

$$\textit{for all } X \in \mathcal{C}, \textit{ the set } \left\{Y \in M^\Phi : Y \succeq_{\mathrm{icv}} X\right\} \textit{ is contained in } \mathcal{C}\,. \tag{6.8}$$

On the other hand, for every subset $\mathcal{B}$ of M^Φ with the properties (6.6)–(6.8)*, $\rho_{\mathcal{B}}$ is an* icv*-monotone monetary risk measure on M^Φ.*

Proof. That the acceptance set of an icv-monotone monetary risk measure satisfies (6.8) is clear. On the other hand, a subset $\mathcal{B}$ of M^Φ with the properties (6.6)–(6.8) also satisfies (6.5). So $\rho_{\mathcal{B}}$ is a monetary risk measure, which obviously is icv-monotone. □

For fixed $X \in M^\Phi$, the set $\left\{Z \in M^\Phi : Z \succeq_{\mathrm{icv}} X\right\}$ is convex. But this is in general not the case for sets of the form $\left\{Z \in M^\Phi : Z \succeq_{\mathrm{icv}} X \text{ or } Z \succeq_{\mathrm{icv}} Y\right\}$ for $X, Y \in M^\Phi$. This allows us to construct the following example of a non-convex icv-monotone monetary risk measure.

Example 6.6. Let $(0,1)$ with the Borel sigma-algebra and the Lebesgue measure be our probability space. Consider the set $\mathcal{B} = \{Z \in L^1 : Z \succeq_{\text{icv}} X \text{ or } Z \succeq_{\text{icv}} Y\}$, where

$$X(x) = \begin{cases} 6x - 1 & 0 < x \leq \frac{1}{3} \\ 6x + 1 & \frac{1}{3} < x < 1 \end{cases} \qquad Y(x) = \begin{cases} 6x + 1 & 0 < x \leq \frac{1}{3} \\ 6x - 1 & \frac{1}{3} < x < 1 \end{cases} .$$

Set $Z(x) = \frac{1}{2}(X+Y)(x) = 6x$. Then $\int_0^1 q^Z(x)dx < \int_0^1 q^X(x)dx$ as well as $\int_0^t q^Z(x)dx < \int_0^t q^Y(x)dx$ for $0 < t < \frac{2}{3}$. Hence, $Z \notin \mathcal{B}$, and $\rho_{\mathcal{B}}$ is a non-convex icv-monotone monetary risk measure on L^1.

6.2 Dual representations of icv-monotone convex monetary risk measures

Average value-at-risk at level $\alpha \in (0,1)$,

$$\text{AVaR}_\alpha(X) := \frac{1}{\alpha}\int_0^\alpha \text{VaR}_y(X)\,dy = -\frac{1}{\alpha}\int_0^\alpha q^X(y)\,dy\,,$$

is a real-valued coherent risk measure on L^1 (see Föllmer and Schied, 2004). (6.2) shows that it is icv-monotone but not strictly icv-monotone. It has been noted before that AVaR can be used as a building block to construct other risk measures; see for instance, Kusuoka (2001), Acerbi (2002, 2004), Föllmer and Schied (2004), Frittelli and Rosazza Gianin (2005), Leitner (2005), Dana (2005), or Jouini et al. (2006). Here, we adapt some of the duality results of Dana (2005) to our setup and combine them with Theorems 2.1 and 2.2 to derive representation results for icv-monotone risk measures on Orlicz hearts. Then we provide characterizations for strict monotonicity, strict convexity modulo translation and strict cv- and icv-monotonicity of icv-monotone risk measures. We are using the following notation:

Definition 6.7. *By $M^\Phi(0,1)$ we denote the Orlicz heart corresponding to Φ over $(0,1)$ equipped with the Borel sigma-algebra and Lebesgue measure. $L^\Psi(0,1)$ is the Orlicz space over $(0,1)$ induced by Ψ. Furthermore, we set*

$$R^\Phi := \left\{q^X : X \in M^\Phi\right\},$$

$$\hat{\mathcal{D}} := \left\{ l : (0,1) \to \mathbb{R}_+ : l \text{ left-continuous, decreasing and } \int_0^1 l(y)dy = 1 \right\},$$

$$\hat{\mathcal{D}}^\Psi := \hat{\mathcal{D}} \cap L^\Psi(0,1) \quad \textit{and} \quad \hat{\mathcal{D}}^\Psi_{\text{s}} := \left\{ l \in \hat{\mathcal{D}}^\Psi : l(y) > 0 \textit{ for all } y \in (0,1) \right\}.$$

If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ over which M^Φ is defined is atomless, it supports a random variable that is uniformly distributed on $(0,1)$, and R^Φ is equal to the convex set $\left\{q^X : X \in M^\Phi(0,1)\right\}$. But if $(\Omega, \mathcal{F}, \mathbb{P})$ has atoms, then R^Φ is smaller than $\left\{q^X : X \in M^\Phi(0,1)\right\}$ and not necessarily convex.

For $l \in \hat{\mathcal{D}}$, define the right-continuous function $\tilde{l} : (0,1] \to \mathbb{R}_+$ by

$$\tilde{l}(y) := \begin{cases} l(y+) & \text{for } y \in (0,1) \\ 0 & \text{for } y = 1 \end{cases} .$$

Then $d\mu(y) = -y d\tilde{l}(y)$ induces a probability measure μ on $(0,1]$ such that

$$l(y) = \int_{[y,1]} \frac{1}{x} d\mu(x) \quad \text{and} \quad \tilde{l}(y) = \int_{(y,1]} \frac{1}{x} d\mu(x) \quad \text{for } y \in (0,1)\,.$$

This provides a bijection between $\hat{\mathcal{D}}$ and the set of probability measures μ on $(0,1]$. For given $l \in \hat{\mathcal{D}}^\Psi$ and $X \in M^\Phi$, one has

$$\left\langle -q^X, l \right\rangle := \int_0^1 -q^X(y) l(y) dy = \int_{(0,1]} \text{AVaR}_y(X) d\mu(y)\,. \tag{6.9}$$

Since (6.9) defines a real-valued coherent risk measure on M^Φ, it follows from Theorem 2.2 that it is continuous in X with respect to $\|.\|_\Phi$. Together with (6.2), (6.9) shows that for $X, Y \in M^\Phi$,

$$X \preceq_{\rm icv} Y \quad \Leftrightarrow \quad \langle q^X, l \rangle \leq \langle q^Y, l \rangle \quad \text{for all } l \in \hat{\mathcal{D}}^\Psi . \tag{6.10}$$

Moreover, for $\xi \in \mathcal{D}^\Psi$, the function l^ξ given by $l^\xi(y) := q^\xi(1-y)$ belongs to $\hat{\mathcal{D}}^\Psi$, and by Hardy–Littlewood's inequality,

$$\left\langle q^X, l^\xi \right\rangle \leq \mathrm{E}_\mathbb{P}[X\xi] \quad \text{for all } X \in M^\Phi ; \tag{6.11}$$

see Hardy et al. (1988) or Föllmer and Schied (2004).

Definition 6.8. *We call a mapping $\nu : \hat{\mathcal{D}}^\Psi \to (-\infty, \infty]$ a penalty function on $\hat{\mathcal{D}}^\Psi$ if it is bounded from below and not identically equal to ∞. We say that it satisfies the growth condition* (G) *if there exist constants $a \in \mathbb{R}$ and $b > 0$ such that*

$$\nu(l) \geq a + b \|l\|_\Psi \quad \text{for all } l \in \hat{\mathcal{D}}^\Psi .$$

The following is a variant of Theorem 2.1 that will be useful to construct examples in Section 8.

Theorem 6.9. *Let ν be a penalty function on $\hat{\mathcal{D}}^\Psi$. Then*

$$\rho_\nu(X) := \sup_{l \in \hat{\mathcal{D}}^\Psi} \left\{ \langle -q^X, l \rangle - \nu(l) \right\}$$

defines a lower semicontinuous icv-monotone convex monetary risk measure on M^Φ, and the implications

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)} \Leftrightarrow \text{(iv)}$$

hold among the conditions:
(i) *ν satisfies the growth condition* (G)
(ii) $\mathrm{core}(\mathrm{dom}\, \rho_\nu) \neq \emptyset$
(iii) *ρ_ν is real-valued and locally Lipschitz-continuous*
(iv) *For each $X \in M^\Phi$ and every sequence $(l_n)_{n \geq 1}$ in $\hat{\mathcal{D}}^\Psi$ satisfying*

$$\lim_{n \to \infty} \left\{ \langle -q^X, l_n \rangle - \nu(l_n) \right\} = \rho_\nu(X) ,$$

the sequences $\langle q^X, l_n \rangle$ and $\nu(l_n)$, $n \geq 1$, are bounded.

If (i) *holds and ν is $(\hat{\mathcal{D}}^\Psi, M^\Phi(0,1))$-lower semicontinuous, then*

$$\rho_\nu(X) = \max_{l \in \hat{\mathcal{D}}^\Psi} \left\{ \langle -q^X, l \rangle - \nu(l) \right\} \quad \text{for all } X \in M^\Phi . \tag{6.12}$$

If the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then the conditions (i)–(iv) *are equivalent.*

Proof. That ρ_ν defines a lower semicontinuous icv-monotone convex monetary risk measure on M^Φ follows from the fact that for every $l \in \hat{\mathcal{D}}^\Psi$, $\langle -q^X, l \rangle$ is a continuous icv-monotone coherent risk measure on M^Φ. By Theorem 2.2, $\rho^\#$ is a penalty function on $\mathcal{D}^\Psi$ with $\rho_\nu = \rho_{\rho^\#}$. So, (ii) $\Leftrightarrow$ (iii) follows from Theorem 2.1. (iv) $\Rightarrow$ (ii) is clear, and (iii) $\Rightarrow$ (iv) can be shown as in the proof of Theorem 2.1. If (i) holds, then the penalty function $\hat{\gamma} : \mathcal{D}^\Psi(0,1) \to (-\infty, \infty]$ given by $\hat{\gamma}(\xi) := \nu(l^\xi)$ satisfies (G), and due to (6.11), one has

$$\hat{\rho}_\nu(X) := \sup_{l \in \hat{\mathcal{D}}^\Psi} \left\{ \langle -q^X, l \rangle - \nu(l) \right\} = \sup_{\xi \in \mathcal{D}^\Psi(0,1)} \left\{ \mathrm{E}_\mathbb{P}[-X\xi] - \hat{\gamma}(\xi) \right\}$$

for all $X \in M^\Phi(0,1)$. So we obtain from Theorem 2.1 that $\hat{\rho}_\nu$ is real-valued. But then also ρ_ν is real-valued. This shows (i) $\Rightarrow$ (ii).

That (i) and $(\hat{\mathcal{D}}^\Psi, M^\Phi(0,1))$-lower semicontinuity of ν imply (6.12) follows as in the proof of Theorem 2.1.

To conclude the proof, assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless. Then it follows from (6.11) that

$$\rho_\nu(X) = \sup_{\xi \in \mathcal{D}^\Psi} \{\mathrm{E}_\mathbb{P}[-X\xi] - \gamma(\xi)\}$$

for the penalty function $\gamma : \mathcal{D}^\Psi \to (-\infty, \infty]$ given by $\gamma(\xi) := \nu(l^\xi)$. By Theorem 2.1, condition (ii) holds if and only if γ satisfies (G), which is equivalent to saying that ν satisfies (G). $\square$

For every convex monetary risk measure ρ on M^Φ, we define

$$\rho^\dagger(l) := \sup_{X \in M^\Phi} \left\{\left\langle -q^X, l\right\rangle - \rho(X)\right\}, \quad l \in \hat{\mathcal{D}}^\Psi.$$

Clearly, $\rho^\dagger$ is lower semicontinuous with respect to $\sigma(\hat{\mathcal{D}}^\Psi, R^\Phi)$ and therefore also with respect to $\sigma(\hat{\mathcal{D}}^\Psi, M^\Phi(0,1))$.

The following is an adaption of Theorem 3.1 in Dana (2005) to our setup:

Theorem 6.10. *Let ρ be a lower semicontinuous convex monetary risk measure on M^Φ. Then the following are equivalent:*

(i) *ρ is icv-monotone*

(ii) $\rho^\#(\xi) = \sup_{X \in M^\Phi} \left\{\left\langle -q^X, l^\xi\right\rangle - \rho(X)\right\}$, $\xi \in \mathcal{D}^\Psi$

(iii) $\rho^\#(\xi) \geq \rho^\#(\xi')$ *for* $\xi, \xi' \in \mathcal{D}^\Psi$ *such that* $\xi \preceq_{\mathrm{cv}} \xi'$

(iv) $\rho(X) = \sup_{\xi \in \mathcal{D}^\Psi} \left\{\left\langle -q^X, l^\xi\right\rangle - \rho^\#(\xi)\right\}$, $X \in M^\Phi$

(v) $\rho(X) = \sup_{l \in \hat{\mathcal{D}}^\Psi} \left\{\left\langle -q^X, l\right\rangle - \rho^\dagger(l)\right\}$, $X \in M^\Phi$

If (i)–(v) *hold, then $\rho^\dagger$ is the smallest penalty function on $\hat{\mathcal{D}}^\Psi$ which induces ρ. If ρ is coherent and* (i)–(v) *hold, then*

$$\rho(X) = \sup_{\xi \in \mathcal{Q}} \left\langle -q^X, l^\xi\right\rangle = \sup_{l \in \mathcal{E}} \left\langle -q^X, l\right\rangle \tag{6.13}$$

for

$$\mathcal{Q} = \left\{\xi \in \mathcal{D}^\Psi : \mathrm{E}_\mathbb{P}[X\xi] + \rho(X) \geq 0 \textit{ for all } X \in M^\Phi\right\}$$

and

$$\mathcal{E} = \left\{l \in \hat{\mathcal{D}}^\Psi : \left\langle q^X, l\right\rangle + \rho(X) \geq 0 \textit{ for all } X \in M^\Phi\right\}.$$

Proof. The equivalence of (i)–(iv) follows as in the proof of Theorem 3.1 in Dana (2005). The implication (v) $\Rightarrow$ (i) is a consequence of Theorem 6.9. On the other hand, if (i)–(iv) hold, one has

$$\rho^\#(\xi) = \rho^\dagger(l^\xi) \quad \text{for all } \xi \in \mathcal{D}^\Psi,$$

and it follows that

$$\rho(X) = \sup_{\xi \in \mathcal{D}^\Psi} \left\{\left\langle -q^X, l^\xi\right\rangle - \rho^\#(\xi)\right\} \leq \sup_{l \in \hat{\mathcal{D}}^\Psi} \left\{\left\langle -q^X, l\right\rangle - \rho^\dagger(l)\right\} \leq \rho(X),$$

which implies (v).

If (i)–(v) hold, then $\rho^\dagger$ must be a penalty function on $\hat{\mathcal{D}}^\Psi$. That it is the smallest one which induces ρ is clear. If ρ is coherent and (i)–(v) hold, then $\rho^\#$ and $\rho^\dagger$ are equal to 0 on the sets $\mathcal{Q}$ and $\mathcal{E}$, respectively, and ∞ otherwise. This shows (6.13). $\square$

Corollary 6.11. *Let $\rho : M^\Phi \to (-\infty, \infty]$ be an icv-monotone convex monetary risk measure with* core(dom ρ) $\neq \emptyset$. *Then*

$$\rho(X) = \max_{\xi \in \mathcal{D}^\Psi} \left\{ \left\langle -q^X, l^\xi \right\rangle - \rho^\#(\xi) \right\} = \max_{l \in \hat{\mathcal{D}}^\Psi} \left\{ \left\langle -q^X, l \right\rangle - \rho^\dagger(l) \right\} , \quad X \in M^\Phi , \tag{6.14}$$

and if ρ is coherent, then

$$\rho(X) = \max_{\xi \in \mathcal{Q}} \left\langle -q^X, l^\xi \right\rangle = \max_{l \in \mathcal{E}} \left\langle -q^X, l \right\rangle , \quad X \in M^\Phi , \tag{6.15}$$

for $\mathcal{Q}$ and $\mathcal{E}$ as in Theorem 6.10.

Proof. By Theorems 2.2 and 6.10, $\rho^\#$ and $\rho^\dagger$ are penalty functions, and

$$\rho(X) = \max_{\xi \in \mathcal{D}^\Psi} \left\{ \mathrm{E}_\mathbb{P}\left[-X\xi\right] - \rho^\#(\xi) \right\} = \sup_{l \in \hat{\mathcal{D}}^\Psi} \left\{ \left\langle -q^X, l \right\rangle - \rho^\dagger(l) \right\} \quad \text{for all } X \in M^\Phi . \tag{6.16}$$

Since for all $\xi \in \mathcal{D}^\Psi$ and $X \in M^\Phi$, one has $\mathrm{E}_\mathbb{P}\left[-X\xi\right] \leq \left\langle -q^X, l^\xi \right\rangle$ by Hardy–Littlewood's inequality (6.11) and $\rho^\#(\xi) = \rho^\dagger(l^\xi)$ by (ii) of Theorem 6.10, the supremum in (6.16) is attained. This shows (6.14). (6.15) follows from (6.14) since for coherent ρ, the penalty functions $\rho^\#$ and $\rho^\dagger$ are equal to 0 on the sets $\mathcal{Q}$ and $\mathcal{E}$, respectively, and ∞ otherwise. □

To characterize properties of icv-monotone risk measures in terms of elements of $\hat{\mathcal{D}}^\Psi$, we need the following definitions:

Definition 6.12. *Let ρ be a distribution-based convex monetary risk measure on M^Φ and ν a penalty function on $\hat{\mathcal{D}}^\Psi$. Then we define the function $\hat{\rho} : R^\Phi \to (-\infty, \infty]$ by $\hat{\rho}(q^X) := \rho(X)$, and we denote*

$$\begin{aligned}
\hat{\chi}_{\hat{\rho}}(r) &:= \left\{ l \in \hat{\mathcal{D}}^\Psi : \hat{\rho}(r) + \rho^\dagger(l) = \langle -r, l \rangle \right\} , \quad r \in R^\Phi \\
\hat{\chi}_{\hat{\rho},\nu}(r) &:= \left\{ l \in \hat{\mathcal{D}}^\Psi : \hat{\rho}(r) + \nu(l) = \langle -r, l \rangle \right\} , \quad r \in R^\Phi \\
\hat{\chi}_{\nu}(l) &:= \left\{ r \in R^\Phi : \hat{\rho}_\nu(r) + \nu(l) = \langle -r, l \rangle \right\} , \quad l \in \hat{\mathcal{D}}^\Psi \\
M^\Phi_\nu &:= \left\{ X \in M^\Phi : \rho_\nu(X) + \nu(l) = \left\langle -q^X, l \right\rangle \text{ for some } l \in \hat{\mathcal{D}}^\Psi \right\} \\
R^\Phi_\nu &:= \left\{ r \in R^\Phi : \hat{\rho}_\nu(r) + \nu(l) = \langle -r, l \rangle \text{ for some } l \in \hat{\mathcal{D}}^\Psi \right\} .
\end{aligned}$$

Theorem 6.13. *Let ν be a penalty function on $\hat{\mathcal{D}}^\Psi$. Then the implications*

$$\text{(i)} \Leftarrow \text{(ii)} \Leftarrow \text{(iii)} \Leftarrow \text{(iv)} \Leftrightarrow \text{(v)}$$

hold among the conditions:

(i) *ρ_ν is strictly monotone on M^Φ_ν*
(ii) *$\hat{\rho}_\nu$ is strictly monotone on R^Φ_ν*
(iii) $\hat{\rho}_\nu(r) = \max_{l \in \hat{\mathcal{D}}^\Psi_\mathrm{s}} \left\{ \langle -r, l \rangle - \nu(l) \right\}$ *for all $r \in R^\Phi_\nu$*
(iv) $\hat{\chi}_{\hat{\rho}_\nu,\nu}(r) \subset \hat{\mathcal{D}}^\Psi_\mathrm{s}$ *for all $r \in R^\Phi_\nu$*
(v) $\hat{\chi}_\nu(l) = \emptyset$ *for all $l \in \hat{\mathcal{D}}^\Psi \setminus \hat{\mathcal{D}}^\Psi_\mathrm{s}$*

If the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms, then all conditions (i)–(v) *are equivalent.*

Proof. (v) ⇔ (iv) ⇒ (iii) ⇒ (ii) follow as the corresponding implications of Theorem 4.2. (ii) ⇒ (i) is clear. To complete the proof it suffices to show (i) ⇒ (iv) when the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms. So assume this is the case and (i) holds but there exist $r \in R^{\Phi}_{\nu}$ and $l \in \hat{\mathcal{D}}^{\Psi} \setminus \hat{\mathcal{D}}^{\Psi}_{\mathrm{s}}$ such that $\hat{\rho}_{\nu}(r) = \langle -r, l \rangle - \nu(l)$. Then there exists $z \in (0,1)$ such that $l(y) = 0$ for $y \in (z,1)$. Choose $X \in M^{\Phi}_{\nu}$ with $q^X = r$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms, there exists a subset $A \subset \left\{X \geq q^X(z)\right\}$ with $\mathbb{P}[A] = 1 - z$. The quantile function of the random variable $Y = X + 1_A$ is equal to $q^X + 1_{[z,1)}$. So

$$\rho_{\nu}(Y) \geq \left\langle -q^Y, l \right\rangle - \nu(l) = \left\langle -q^X, l \right\rangle - \nu(l) = \rho_{\nu}(X) \geq \rho_{\nu}(Y) \,.$$

But this implies $Y \in M^{\Phi}_{\nu}$ and $\rho_{\nu}(Y) = \rho_{\nu}(X)$, a contradiction to (i). □

Theorem 6.14. *For a penalty function ν on $\hat{\mathcal{D}}^{\Psi}$, the implications*

$$\text{(i)}, \text{(ii)} \Leftarrow \text{(iii)} \Leftarrow \text{(iv)} \Leftrightarrow \text{(v)}$$

hold among the conditions

(i) *ρ_{ν} is strictly convex modulo translation on M^{Φ}_{ν}*

(ii) *$\hat{\rho}_{\nu}$ is strictly convex modulo translation on R^{Φ}_{ν}*

(iii) *$\hat{\chi}_{\hat{\rho}_{\nu},\nu}(r) \setminus \hat{\chi}_{\hat{\rho}_{\nu},\nu}(s) \neq \emptyset$ for all $r, s \in R^{\Phi}_{\nu}$ such that $r \not\sim_{\mathrm{t}} s$*

(iv) *$\hat{\chi}_{\hat{\rho}_{\nu},\nu}(r) \cap \hat{\chi}_{\hat{\rho}_{\nu},\nu}(s) = \emptyset$ for all $r, s \in R^{\Phi}_{\nu}$ such that $r \not\sim_{\mathrm{t}} s$*

(v) *for all $l \in \hat{\mathcal{D}}^{\Psi}$, $\hat{\chi}_{\nu}(l)$ contains at most one element modulo translation in R^{Φ}.*

If the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms, then the conditions (i)–(v) *are equivalent.*

Proof. The implications (v) ⇔ (iv) ⇒ (iii) are obvious. (iii) ⇒ (ii) follows as the implication (ii) ⇒ (i) of Theorem 5.3. To prove (iii) ⇒ (i), assume there exist $X, Y \in M^{\Phi}_{\nu}$ and $\lambda \in (0,1)$ such that $X \not\sim_{\mathrm{t}} Y$ and $\lambda X + (1-\lambda) Y \in M^{\Phi}_{\nu}$. By Lemma 6.15 below, we have $q^{\lambda X + (1-\lambda)Y} \not\sim_{\mathrm{t}} q^X$ or $q^{\lambda X + (1-\lambda)Y} \not\sim_{\mathrm{t}} q^Y$. Therefore, there exists $l \in \hat{\chi}_{\hat{\rho}_{\nu},\nu}(q^{\lambda X + (1-\lambda)Y})$ which does not belong to $\hat{\chi}_{\hat{\rho}_{\nu},\nu}(q^X) \cap \hat{\chi}_{\hat{\rho}_{\nu},\nu}(q^Y)$, and we get

$$\begin{aligned} & \rho_{\nu}(\lambda X + (1-\lambda)Y) = \left\langle -q^{\lambda X + (1-\lambda)Y}, l \right\rangle - \nu(l) \\ \leq \;& \lambda \left\langle -q^X, l \right\rangle + (1-\lambda) \left\langle -q^Y, l \right\rangle - \nu(l) < \lambda \rho_{\nu}(X) + (1-\lambda) \rho_{\nu}(Y) \,. \end{aligned}$$

If the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms, it supports a random variable U that is uniformly distributed on $(0,1)$. Then the mapping $r \mapsto r(U)$ embeds R^{Φ}_{ν} in M^{Φ}_{ν}, and (i) implies (ii). Moreover, R^{Φ} is convex, and (ii) ⇒ (iv) follows as the implication (i) ⇒ (iii) of Theorem 5.3. □

Lemma 6.15. *Let $X, Y \in L^1$ and $\lambda \in (0,1)$ such that $q^X \sim_{\mathrm{t}} q^{\lambda X + (1-\lambda)Y} \sim_{\mathrm{t}} q^Y$. Then $X \sim_{\mathrm{t}} Y$.*

Proof. Denote $Z = Y + \mathrm{E}_{\mathbb{P}}[X - Y]$. Then

$$q^X \sim_{\mathrm{t}} q^{\lambda X + (1-\lambda)Z} \sim_{\mathrm{t}} q^Z \,, \tag{6.17}$$

and $\mathrm{E}_{\mathbb{P}}[X] = \mathrm{E}_{\mathbb{P}}[\lambda X + (1-\lambda)Z] = \mathrm{E}_{\mathbb{P}}[Z]$, which is equivalent to

$$\int_0^1 q^X(y) dy = \int_0^1 q^{\lambda X + (1-\lambda)Z}(y) dy = \int_0^1 q^Z(y) dy \,. \tag{6.18}$$

(6.17) and (6.18) imply $q^X = q^{\lambda X + (1-\lambda)Z} = q^Z$. So one has

$$\mathrm{E}_{\mathbb{P}}[f(X)] = \mathrm{E}_{\mathbb{P}}[f(\lambda X + (1-\lambda)Z)] \geq \lambda \mathrm{E}_{\mathbb{P}}[f(X)] + (1-\lambda) \mathrm{E}_{\mathbb{P}}[f(Z)] = \mathrm{E}_{\mathbb{P}}[f(X)]$$

for all concave functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(X) \in L^1$. This shows that $X = Z$ and hence, $X \sim_{\mathrm{t}} Y$. □

Theorem 6.16. *Let ν be a penalty function on $\hat{\mathcal{D}}^\Psi$. Then ρ_ν is strictly* cv*-monotone on M^Φ_ν if and only if for all $l \in \hat{\mathcal{D}}^\Psi$ and $r \in \hat{\chi}_\nu(l)$, r is a deterministic function of the right-continuous function $\tilde{l}(y) = l(y+)$.*

Proof. To show the "only if"-direction, assume that ρ_ν is strictly cv-monotone on M^Φ_ν but there exist $l \in \hat{\mathcal{D}}^\Psi$ and $r \in \hat{\chi}_\nu(l)$ such that r is not a deterministic function of $\tilde{l}$. Then r cannot be $\sigma(\tilde{l})$-measurable, and therefore, $r \prec_{\mathrm{cv}} \mathrm{E}\left[r \mid \tilde{l}\right]$. Thus

$$\rho^\dagger_\nu(l) \le \nu(l) = \langle -r, l\rangle - \hat{\rho}_\nu(r) \le \left\langle -\mathrm{E}\left[r \mid \tilde{l}\right], l\right\rangle - \hat{\rho}_\nu\left(\mathrm{E}\left[r \mid \tilde{l}\right]\right) \le \rho^\dagger_\nu(l)\,.$$

But this implies $\mathrm{E}\left[r \mid \tilde{l}\right] \in \hat{\chi}_\nu(l)$ and therefore, $\hat{\rho}_\nu(r) > \hat{\rho}_\nu\left(\mathrm{E}\left[r \mid \tilde{l}\right]\right)$, a contradiction.

For the "if"-part, assume ρ_ν is not strictly cv-monotone on M^Φ_ν. Then there exist $r, s \in R^\Phi_\nu$ such that $r \prec_{\mathrm{cv}} s$ and $\hat{\rho}_\nu(r) \le \hat{\rho}_\nu(s)$. Choose $l \in \hat{\chi}_{\hat{\rho}_\nu,\nu}(s)$ and observe that

$$\rho^\dagger_\nu(l) \le \nu(l) = \langle -s, l\rangle - \hat{\rho}_\nu(s) \le \langle -r, l\rangle - \hat{\rho}_\nu(r) \le \rho^\dagger_\nu(l)\,.$$

It follows that $r \in \hat{\chi}_\nu(l)$ and $\langle r, l\rangle = \langle s, l\rangle$. Since $r \prec_{\mathrm{cv}} s$, the continuous function $f(z) := \int_0^z s(y) - r(y)dy$ is non-negative and satisfies $f(0) = f(1) = 0$ as well as $\max_{0 \le z \le 1} f(z) > 0$. Let $z_0 \in (0,1)$ be a maximizer of f and denote

$$z_1 := \max\left\{z \le z_0 : f(z) = f(z_0)/2\right\} \quad \text{and} \quad z_2 := \min\left\{z \ge z_0 : f(z) = f(z_0)/2\right\}\,.$$

Then there exist $z_3 \in [z_1, z_0]$ and $z_4 \in [z_0, z_2]$ such that $s(z_3) > r(z_3)$ and $s(z_4) < r(z_4)$. Since s is increasing, this implies $r(z_3) < r(z_4)$. But due to $\langle r, l\rangle = \langle s, l\rangle$, we have

$$\int_0^1 f(y)d\tilde{l}(y) = -\int_0^1 \tilde{l}(y)df(y) = \int_0^1 \tilde{l}(y)(r(y) - s(y))dy = 0\,,$$

and it follows that $\tilde{l}(z_1) = \tilde{l}(z_3) = \tilde{l}(z_4) = \tilde{l}(z_2)$. So r cannot be a deterministic function of $\tilde{l}$. □

Remark 6.17. Lemma 2.3 of Dana (2005) extended to Orlicz hearts yields that for fixed $Y \in M^\Phi$,

$$\left\{X \in M^\Phi : X \succeq_{\mathrm{icv}} Y\right\} = \left\{X \in M^\Phi : X \succeq_{\mathrm{cv}} Y\right\} + M^\Phi_+\,.$$

This shows that an icv-monotone monetary risk measure ρ on M^Φ is strictly icv-monotone on M^Φ if and only if ρ is strictly monotone and strictly cv-monotone on M^Φ.

7 Cash-additive hulls

Let V be a mapping from M^Φ to $(-\infty, \infty]$ satisfying the following three properties:

(V1) $V(X) \le V(Y)$ for all $X, Y \in M^\Phi$ such that $X \le Y$
(V2) $V(\lambda X + (1-\lambda)Y) \le \lambda V(X) + (1-\lambda)V(Y)$ for all $X, Y \in M^\Phi$ and $\lambda \in (0,1)$
(V3) for all $X \in M^\Phi$, $\inf_{s \in \mathbb{R}}\{V(s - X) - s\} \in \mathbb{R}$ and the infimum is attained.

Then

$$\rho^V(X) := \min_{s \in \mathbb{R}}\{V(s - X) - s\}$$

is the largest real-valued convex monetary risk measure on M^Φ such that

$$\rho^V(X) \le V(-X) \quad \text{for all } X \in M^\Phi\,.$$

We call it the cash-additive hull of the the decreasing convex functional $V(-.)$; see Section 5.1 of Cheridito and Li (2007).

Proposition 7.1. *Let $X \in M^\Phi$ and $s_X \in \mathbb{R}$ such that $\rho^V(X) = V(s_X - X) - s_X$. If V is Gâteaux-differentiable at $s_X - X$, then ρ^V is Gâteaux-differentiable at X with*

$$\nabla \rho^V(X) = -\nabla V(s_X - X)\,.$$

Proof. If V is Gâteaux-differentiable at $s_X - X$, then

$$\begin{aligned}
\left(\rho^V\right)'(X;Y) &= \lim_{\varepsilon \downarrow 0} \frac{\rho^V(X+\varepsilon Y) - \rho^V(X)}{\varepsilon} \\
&\leq \lim_{\varepsilon \downarrow 0} \frac{V(s_X - X - \varepsilon Y) - V(s_X - X)}{\varepsilon} \\
&= V'(s_X - X; -Y) = \mathrm{E}_{\mathbb{P}}\left[-Y \nabla V(s_X - X)\right]
\end{aligned}$$

for all $Y \in M^\Phi$. Since $\left(\rho^V\right)'(X;.)$ is sublinear, one also has

$$\left(\rho^V\right)'(X;Y) \geq -\left(\rho^V\right)'(X;-Y) \geq \mathrm{E}_{\mathbb{P}}\left[-Y \nabla V(s_X - X)\right]\,,$$

and it follows that ρ^V is Gâteaux-differentiable at X with $\nabla \rho^V(X) = -\nabla V(s_X - X)$. □

Proposition 7.2. *If V is strictly monotone on $\operatorname{dom} V$, then ρ^V is strictly monotone on M^Φ.*

Proof. Let $X, Y \in M^\Phi$ with $X \leq Y$ and $\mathbb{P}[X < Y] > 0$. Then there exists $s_X \in \mathbb{R}$ such that

$$\rho^V(X) = V(s_X - X) - s_X > V(s_X - Y) - s_X \geq \rho^V(Y)\,.$$

□

Proposition 7.3. *If V is strictly convex modulo translation (comonotonicity) on $\operatorname{dom} V$, then ρ^V is strictly convex modulo translation (comonotonicity) on M^Φ.*

Proof. Let $X, Y \in M^\Phi$, such that $X \not\sim_{\mathrm{t}} Y$ $(X \not\sim_{\mathrm{c}} Y)$ and $\lambda \in (0,1)$. There exist $s_X, s_Y \in \mathbb{R}$ such that

$$\rho^V(X) = V(s_X - X) - s_X \quad \text{and} \quad \rho^V(Y) = V(s_Y - Y) - s_Y\,.$$

Then $s_X - X \not\sim_{\mathrm{t}} s_Y - Y$ $(s_X - X \not\sim_{\mathrm{c}} s_Y - Y)$, and therefore

$$\begin{aligned}
& \lambda \rho^V(X) + (1-\lambda)\rho^V(Y) \\
= {} & \lambda \left\{V(s_X - X) - s_X\right\} + (1-\lambda)\left\{V(s_Y - Y) - s_Y\right\} \\
> {} & V(\lambda s_X + (1-\lambda)s_Y - [\lambda X + (1-\lambda)Y]) - [\lambda s_X + (1-\lambda)s_Y] \\
\geq {} & \rho^V(\lambda X + (1-\lambda)Y)\,.
\end{aligned}$$

□

Proposition 7.4. *If V is (strictly)* icx*-monotone on $\operatorname{dom} V$, then ρ^V is (strictly)* icv*-monotone on M^Φ. If V is distribution-based, then so is ρ^V.*

Proof. Assume V is icx-monotone on $\operatorname{dom} V$ and $X \preceq_{\mathrm{icv}} Y$. Then $-X \succeq_{\mathrm{icx}} -Y$, and there exists $s_X \in \mathbb{R}$ such that

$$\rho^V(X) = V(s_X - X) - s_X \geq V(s_X - Y) - s_X \geq \rho^V(Y)\,.$$

This shows that ρ^V is icv-monotone on M^ϕ. The other claims follow analogously. □

8 Examples

8.1 Transformed loss risk measures

Let $H : \mathbb{R} \to \mathbb{R}$ be an increasing convex function with the property

$$\lim_{|x|\to\infty} \{H(x) - x\} = \infty .$$

Then

$$V(X) = \mathrm{E}_{\mathbb{P}}[H(X)] \tag{8.1}$$

is a real-valued mapping on the Orlicz heart M^Φ corresponding to the function $\Phi(x) := H(x) - H(0)$. It clearly satisfies (V1)–(V3) and is icx-monotone. So, by Proposition 7.4, ρ^V is a real-valued icv-monotone convex monetary risk measure on M^Φ. Its minimal penalty function is given by

$$\left(\rho^V\right)^{\#}(\mathbb{Q}) = \mathrm{E}_{\mathbb{P}}\left[H^*\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad \mathbb{Q} \in \mathcal{D}^\Psi ; \tag{8.2}$$

see Section 5.4 of Cheridito and Li (2007). If H is strictly increasing, then V is strictly monotone on M^Φ, which by Proposition 7.2 implies that also ρ^V is strictly monotone on M^Φ. If H is strictly convex, then V is strictly convex and strictly icx-monotone on M^Φ, and so by Propositions 7.3 and 7.4, ρ^V is strictly convex modulo translation and strictly icv-monotone on M^Φ. If H is differentiable, then V is Gâteaux-differentiable on M^Φ with $\nabla V(X) = H'(X)$ and it follows from Proposition 7.1 that ρ^V is Gâteaux-differentiable on M^Φ with $\nabla\rho^V(X) = -H'(s_X - X)$ for $s_X \in \mathbb{R}$ such that $\rho^V(X) = V(s_X - X) - s_X$.

For $H^*(1) = 0$, (8.2) is an f-divergence after Csiszar (1967) and can be interpreted as a distance between $\mathbb{Q}$ and $\mathbb{P}$. Functionals of the form ρ^V for V equal to (8.1) have appeared in different settings in Ben-Tal and Teboulle (1987), Schied (2007), Cheridito and Li (2007), Cherny and Kupper (2007).

8.2 Transformed norm risk measures

Let F be a left-continuous increasing convex function from $[0,\infty)$ to $(-\infty,\infty]$ such that $\lim_{x\to\infty} F(x) = \infty$, $G : [0,\infty) \to [0,\infty)$ a convex function with $G(0) = 0$ and $\lim_{x\to\infty} G(x) = \infty$, and $H : \mathbb{R} \to [0,\infty)$ an increasing convex function with $\lim_{x\to\infty} H(x) = \infty$. Assume the following two conditions hold:

$$\text{(FGH1)} \qquad F\left(\frac{H(x)+\varepsilon}{G^{-1}(1)}\right) < \infty \quad \text{for some } x \in \mathbb{R} \text{ and } \varepsilon > 0$$

$$\text{(FGH2)} \qquad \lim_{x\to\infty} \left\{F \circ H(x) - G^{-1}(1)\,x\right\} = \infty .$$

Define $H_0(x) := H(x) - H(0)$ for $x \geq 0$. Then $\Phi := G \circ H_0$ is a convex function from $[0,\infty)$ to $[0,\infty)$ with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$. In Section 5.2 of Cheridito and Li (2007) it is shown that $V(X) = F\left(\|H(X)\|_G\right)$ is a well-defined mapping from M^Φ to $(-\infty,\infty]$ satisfying (V1)–(V3). It can easily be checked that it is icx-monotone. So it follows from Proposition 7.4 that ρ^V defines a real-valued icv-monotone convex monetary risk measure on M^Φ. Its minimal penalty function is given in Theorem 5.3 of Cheridito and Li (2007).

Clearly, the Luxemburg norm $\|.\|_\Phi$ is strictly monotone on M^Φ_+ if and only if Φ is strictly increasing. So if F, G, H are strictly increasing, then V is strictly monotone, and it follows from Proposition 7.2 that the same is true for ρ^V. If F and G are strictly increasing and H is strictly convex, then V is strictly convex and strictly icx-monotone, and so by Propositions 7.3 and 7.4, ρ^V is strictly convex modulo translation and strictly icv-monotone.

As a specific example, consider the risk measure

$$\rho(X) := \min_{s \in \mathbb{R}} \left\{ \frac{1}{\alpha} \left\| (s-X)^+ \right\|_p^\beta - s \right\} \tag{8.3}$$

for (α, β, p) in $(0,1) \times \{1\} \times [1,\infty)$ or $(0,\infty) \times (1,\infty) \times [1,\infty)$. ρ is real-valued on L^p, and if $s_X \in \mathbb{R}$ minimizes the right side of (8.3), then $s_X \geq \operatorname{ess\,inf} X$. Moreover, for $\beta > 1$, s_X is unique, $s_X > \operatorname{ess\,inf} X$, and the minimal penalty function of ρ is given by

$$\rho^{\#}(\mathbb{Q}) = c \left\| \mathbb{Q} \right\|_q^d \quad \text{for } q := \frac{p}{p-1}\,,\; d := \frac{\beta}{\beta-1}\,,\; c := \alpha^{d-1} \beta^{1-d} d^{-1}\,. \tag{8.4}$$

For $\beta = 1$, ρ is coherent, s_X is not necessarily unique, and

$$\rho^{\#}(\mathbb{Q}) = \begin{cases} 0 & \text{if } \left\| \mathbb{Q} \right\|_q \leq \frac{1}{\alpha} \\ \infty & \text{if } \left\| \mathbb{Q} \right\|_q > \frac{1}{\alpha} \end{cases} \quad \text{for } q = \frac{p}{p-1} \tag{8.5}$$

(for proofs of (8.4) and (8.5), see Section 5.3 of Cheridito and Li, 2007).

If $\beta, p > 1$, then $V(X) = \frac{1}{\alpha} \left\| X^+ \right\|_p^\beta$ is Gâteaux-differentiable on L^p with

$$\nabla V(X) = \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}} \left[(X^+)^p \right]^{\frac{\beta}{p}-1} \left(X^+ \right)^{p-1}.$$

Hence, it follows from Proposition 7.1 that ρ is Gâteaux-differentiable on L^p with

$$\nabla \rho(X) = -\frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}} \left[((s_X - X)^+)^p \right]^{\frac{\beta}{p}-1} \left((s_X - X)^+ \right)^{p-1}.$$

By Proposition 3.2, $\nabla \rho(X)$ is in $-\mathcal{D}^\Psi$. So it can be written as

$$\nabla \rho(X) = -\frac{((s_X - X)^+)^{p-1}}{\mathrm{E}\left[((s_X - X)^+)^{p-1} \right]}\,. \tag{8.6}$$

For $\beta = 1$, $p > 1$ and $X \in L^p$ with $\mathbb{P}\left[X = \operatorname{ess\,inf} X \right] < \alpha^p$, one easily checks that $\mathbb{P}\left[X < s_X \right] > 0$. Hence, $V(.) = \frac{1}{\alpha} \left\| (.)^+ \right\|_p$ is Gâteaux-differentiable at $s_X - X$ with

$$\nabla V(s_X - X) = \frac{1}{\alpha} \mathrm{E}_{\mathbb{P}} \left[((s_X - X)^+)^p \right]^{\frac{1}{p}-1} \left((s_X - X)^+ \right)^{p-1},$$

and it follows from Proposition 7.1 that ρ is Gâteaux-differentiable at X with Gâteaux-derivative (8.6).

If $\beta = 1$, $p \geq 1$ and $X \in L^p$ such that $\mathbb{P}\left[X = \operatorname{ess\,inf} X \right] \geq \alpha^p$, then the measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1_{\{X = \operatorname{ess\,inf} X\}}}{\mathbb{P}\left[X = \operatorname{ess\,inf} X \right]}$$

satisfies

$$\left\| \mathbb{Q} \right\|_q \leq \frac{1}{\alpha} \quad \text{and} \quad \mathrm{E}_{\mathbb{Q}}\left[-X \right] = -\operatorname{ess\,inf} X\,.$$

So $\mathbb{Q}$ is a maximizer of the right side of

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{D}^q,\, \left\| \mathbb{Q} \right\|_q \leq 1/\alpha} \mathrm{E}_{\mathbb{Q}}\left[-X \right],$$

but not necessarily the only one.

For $\beta \geq 1$, $p = 1$ and $X \in L^1$ with $\mathbb{P}[X = s_X] = 0$, $V(.) = \frac{1}{\alpha} \mathrm{E}_{\mathbb{P}}[(.)^+]^\beta$ is Gâteaux-differentiable at $s_X - X$ with

$$\nabla V(s_X - X) = \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} 1_{\{X < s_X\}} .$$

So it follows from Proposition 7.1 that ρ is Gâteaux-differentiable at X with

$$\nabla \rho(X) = -\frac{1_{\{X < s_X\}}}{\mathbb{P}[X < s_X]} .$$

If $\beta > 1$, $p = 1$ and $X \in L^1$ with $\mathbb{P}[X = s_X] > 0$, the left- and right-derivative of the function $s \mapsto \frac{1}{\alpha} \mathrm{E}_{\mathbb{P}}[(s - X)^+]^\beta - s$ at s_X are given by

$$\frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} \mathbb{P}[X < s_X] - 1 \leq 0 \quad \text{and} \quad \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} \mathbb{P}[X \leq s_X] - 1 \geq 0 ,$$

respectively. Choose any random variable ζ such that

$$0 \leq \zeta \leq \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} 1_{\{X = s_X\}} \quad \text{and} \quad \mathrm{E}_{\mathbb{P}}[\zeta] = 1 - \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} \mathbb{P}[X < s_X] .$$

Then

$$\xi = \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} 1_{\{X < s_X\}} + \zeta$$

is the density of a probability measure $\mathbb{Q}$ such that

$$\begin{aligned}
& \mathrm{E}_{\mathbb{Q}}[-X] - \alpha^{d-1} \beta^{1-d} d^{-1} \|\mathbb{Q}\|_\infty^d \\
= \; & \mathrm{E}_{\mathbb{Q}}\left[(s_X - X)^+\right] - \frac{\beta}{\alpha d} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta d - d} - s_X \\
= \; & \frac{\beta}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^{\beta-1} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right] - \frac{\beta}{\alpha d} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^\beta - s_X \\
= \; & \frac{1}{\alpha} \mathrm{E}_{\mathbb{P}}\left[(s_X - X)^+\right]^\beta - s_X = \rho(X) .
\end{aligned}$$

Thus, it follows from (8.4) that $\mathbb{Q}$ maximizes the right side of

$$\rho(X) = \max_{\mathbb{Q} \in \mathcal{D}^\infty} \left\{ \mathrm{E}_{\mathbb{Q}}[-X] - \alpha^{d-1} \beta^{1-d} d^{-1} \|\mathbb{Q}\|_\infty^d \right\} .$$

But it is not necessarily the only measure in $\mathcal{D}^\infty$ with this property.

For $\beta = p = 1$ and $X \in L^1$ such that $\mathbb{P}[X = s_X] > 0$, it is well known that the maximizing measures for ρ at X are of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{1}{\alpha} 1_{\{X < s_X\}} + \zeta ,$$

where ζ is a random variable satisfying

$$0 \leq \zeta \leq \frac{1}{\alpha} 1_{\{X = s_X\}} \quad \text{and} \quad \mathrm{E}_{\mathbb{P}}[\zeta] = 1 - \frac{1}{\alpha} \mathbb{P}[X < s_X]$$

(see, Cherny, 2006).

Since $\chi_\rho(X) \subset \mathcal{D}^q_{\mathrm{s}}$ does in general not hold for risk measures of the form (8.3), it follows from Theorem 4.2, that they are not strictly monotone on L^p and hence not strictly convex modulo translation by Proposition 5.2.

8.3 Delta spectral measures

Proposition 8.1. *Let $p \in [1,\infty)$ and $\eta : (0,1] \to (-\infty,\infty]$ a function that is not identically equal to ∞. If there exist constants $a \in \mathbb{R}$ and $b > 0$ such that*

$$\eta(\lambda) \geq a + b\lambda^{-1/p} \quad \textit{for all } \lambda \in (0,1]\,, \tag{8.7}$$

then

$$\rho(X) = \sup_{\lambda \in (0,1]} \left\{ \mathrm{AVaR}_\lambda(X) - \eta(\lambda) \right\} \tag{8.8}$$

defines a real-valued locally Lipschitz-continuous icv*-monotone convex monetary risk measure on L^p. If η satisfies* (8.7) *and is lower semicontinuous, then the supremum in* (8.8) *is attained. On the other hand, if the underlying probability space is atomless and* (8.8) *is finite for all $X \in L^p$, then* (8.7) *must hold.*

Proof. For each $\lambda \in (0,1]$, $\mathrm{AVaR}_\lambda(X)$ can be written as $\langle -q^X, l_\lambda \rangle$ for

$$l_\lambda(y) = \lambda^{-1} 1_{(0,\lambda]}(y) \in \hat{\mathcal{D}}\,.$$

Set $q = p/(p-1)$ and define the function $\nu : \hat{\mathcal{D}}^q \to (-\infty,\infty]$ by

$$\nu(l) := \begin{cases} \eta(\lambda) & \text{if } l = l_\lambda \text{ for some } \lambda \in (0,1] \\ \infty & \text{else} \end{cases}\,.$$

Since $\|l_\lambda\|_q = \lambda^{-1/p}$, the mapping ν satisfies the growth condition (G) if and only if η fulfills (8.7). Moreover, $\sigma(\hat{\mathcal{D}}^q, L^p(0,1))$-lower semicontinuity of ν is equivalent to lower semicontinuity of η. Hence the proposition follows from Theorem 6.9. □

Example 8.2. For $\alpha > 0$ and $p \in [1,\infty)$, $\eta(\lambda) = \alpha\lambda^{-1/p}$ satisfies (8.7) and is continuous on $(0,1]$. So by Proposition 8.1,

$$\rho(X) = \max_{\lambda \in (0,1]} \left\{ \mathrm{AVaR}_\lambda(X) - \alpha\lambda^{-1/p} \right\} \tag{8.9}$$

defines a real-valued locally Lipschitz-continuous icv-monotone convex monetary risk measure on L^p.

If $\mathrm{VaR}_\lambda(X)$ is continuous in λ, then the maximum in (8.9) is either attained at $\lambda = 1$ or at $\lambda = \lambda_0$ such that

$$\frac{d}{d\lambda} \left(\mathrm{AVaR}_\lambda(X) - \alpha\lambda^{-1/p} \right) \Big|_{\lambda=\lambda_0} = 0\,,$$

or equivalently,

$$\mathrm{AVaR}_{\lambda_0}(X) - \mathrm{VaR}_{\lambda_0}(X) = \frac{\alpha}{p}\lambda_0^{-1/p}\,.$$

Since $l_\lambda \notin \hat{\mathcal{D}}^q_s$ for $\lambda \in (0,1)$, it follows from Theorem 6.13 that ρ is in general not strictly monotone. By Proposition 5.2, it is not strictly convex modulo translation either.

8.4 Uniform spectral measures

Proposition 8.3. *Let $p \in (1,\infty)$ and $\eta : (0,1] \to (-\infty,\infty]$ a function which is not identically equal to ∞. If there exist constants $a \in \mathbb{R}$ and $b > 0$ such that*

$$\eta(\lambda) \geq a + b\lambda^{-1/p} \quad \textit{for all } \lambda \in (0,1]\,, \tag{8.10}$$

then

$$\rho(X) = \sup_{\lambda \in (0,1]} \left\{ \frac{1}{\lambda} \int_0^\lambda \text{AVaR}_y(X) dy - \eta(\lambda) \right\} \tag{8.11}$$

defines a real-valued locally Lipschitz-continuous icv*-monotone convex monetary risk measure on L^p. If η satisfies* (8.10) *and is lower semicontinuous, then the supremum in* (8.11) *is attained. On the other hand, if the underlying probability space is atomless and* (8.11) *is finite for all $X \in L^p$, then* (8.10) *must hold.*

Proof. By (6.9), one has for all $\lambda \in (0,1]$,

$$\frac{1}{\lambda} \int_0^\lambda \text{AVaR}_y(X) dy = \left\langle -q^X, l_\lambda \right\rangle ,$$

where

$$l_\lambda(y) = \begin{cases} \frac{1}{\lambda} \log\left(\frac{\lambda}{y}\right) & \text{for } y \le \lambda \\ 0 & \text{for } y > \lambda \end{cases} .$$

Set $q = p/(p-1)$ and define the function $\nu : \hat{\mathcal{D}}^q \to (-\infty, \infty]$ by

$$\nu(l) := \begin{cases} \eta(\lambda) & \text{if } l = l_\lambda \text{ for some } \lambda \in (0,1] \\ \infty & \text{else} \end{cases} .$$

With the change of variables $x = \log(\lambda/y)$, one obtains

$$\int_0^\lambda \log^q\left(\frac{\lambda}{y}\right) dy = \int_0^\infty \lambda x^q e^{-x} dx = \lambda \Gamma(q+1) .$$

So

$$\|l_\lambda\|_q = \Gamma(q+1)^{1/q} \lambda^{1/q-1} = \Gamma(q+1)^{1/q} \lambda^{-1/p} ,$$

and the proposition follows from Theorem 6.9 like Proposition 8.1. □

Example 8.4. For $\alpha > 0$ and $p \in (1, \infty)$, the function $\eta(\lambda) = \alpha \lambda^{-1/p}$ satisfies (8.10) and is continuous on $(0,1]$. So it follows from Proposition 8.3 that

$$\rho(X) = \max_{\lambda \in (0,1]} \left\{ \frac{1}{\lambda} \int_0^\lambda \text{AVaR}_y(X) dy - \alpha \lambda^{-1/p} \right\} \tag{8.12}$$

is a real-valued locally Lipschitz-continuous icv-monotone convex monetary risk measure on L^p.

Since for each $X \in L^p$, $\text{AVaR}_\lambda(X)$ is continuous in λ, the maximum in (8.12) is either attained at $\lambda = 1$ or $\lambda = \lambda_0$ satisfying

$$\frac{d}{d\lambda} \left(\frac{1}{\lambda} \int_0^\lambda \text{AVaR}_y(X) dy - \alpha \lambda^{-1/p} \right) \Big|_{\lambda = \lambda_0} = 0 ,$$

or

$$\frac{1}{\lambda_0} \int_0^{\lambda_0} \text{AVaR}_y(X)\, dy - \text{AVaR}_{\lambda_0}(X) = \frac{\alpha}{p} \lambda_0^{-1/p} .$$

Again, $l_\lambda \notin \hat{\mathcal{D}}^q_{\text{s}}$ for $\lambda \in (0,1)$. Hence, by Theorem 6.13, ρ is in general not strictly monotone, and by Proposition 5.2, not strictly convex modulo translation either.

8.5 Power spectral measures

Proposition 8.5. *Let $p \in (1,\infty)$, $q = p/(p-1)$ and η a mapping from $[0,1/q)$ to $(-\infty,\infty]$ that is not identically equal to ∞. If there exist constants $a \in \mathbb{R}$ and $b > 0$ such that*

$$\eta(\lambda) \geq a + b(1-q\lambda)^{-1/q} \quad \text{for all } \lambda \in [0,1/q)\,, \tag{8.13}$$

then

$$\rho(X) = \sup_{\lambda \in [0,1/q)} \left\{ \int_0^1 \mathrm{AVaR}_y(X)(1-\lambda)y^{-\lambda}dy - \eta(\lambda) \right\} \tag{8.14}$$

is a real-valued locally Lipschitz-continuous icv*-monotone convex monetary risk measure on L^p. If η satisfies* (8.13) *and is lower semicontinuous, then the supremum in* (8.14) *is attained. On the other hand, if the underlying probability space is atomless and* (8.14) *is finite for all $X \in L^p$, then* (8.13) *must hold.*

Proof. By (6.9), one has

$$\int_0^1 \mathrm{AVaR}_y(X)(1-\lambda)y^{-\lambda}dy = \left\langle -q^X, l_\lambda \right\rangle\,,$$

for all $\lambda \in [0,1)$, where

$$l_0(y) = \log\left(\frac{1}{y}\right) \quad \text{and} \quad l_\lambda(y) = \frac{1-\lambda}{\lambda}(y^{-\lambda}-1) \qquad \text{for } \lambda \in (0,1)\,.$$

It can easily be checked that the mapping $\lambda \mapsto \|l_\lambda\|_q$ is continuous on $[0,1/q)$, and for $\lambda \uparrow 1/q$, one has

$$\frac{1-\lambda}{\lambda}\left[(1-q\lambda)^{-1/q}-1\right] = \frac{1-\lambda}{\lambda}\left(\left\|y^{-\lambda}\right\|_q - 1\right) \leq \|l_\lambda\|_q \leq \frac{1-\lambda}{\lambda}\left\|y^{-\lambda}\right\|_q = \frac{1-\lambda}{\lambda}(1-q\lambda)^{-1/q}\,.$$

So the proposition follows from Proposition 6.9 like Propositions 8.1 and 8.3. □

Example 8.6. Let $p \in (1,\infty)$, $q = p/(p-1)$, $\alpha > 0$ and $\beta \geq 1/q$. Then $\eta(\lambda) = \alpha(1-q\lambda)^{-\beta}$ satisfies (8.13) and is continuous on $[0,1/q)$. Hence, it follows from Proposition 8.5 that

$$\rho(X) = \max_{\lambda \in [0,1/q)} \left\{ \int_0^1 \mathrm{AVaR}_y(X)(1-\lambda)y^{-\lambda}dy - \alpha(1-q\lambda)^{-\beta} \right\} \tag{8.15}$$

is a real-valued locally Lipschitz-continuous icv-monotone convex monetary risk measure on L^p.

The maximum in (8.15) is either attained at $\lambda = 0$ or $\lambda = \lambda_0$ satisfying

$$\frac{d}{d\lambda}\left(\int_0^1 \mathrm{AVaR}_y(X)(1-\lambda)y^{-\lambda}dy - \alpha(1-q\lambda)^{-\beta}\right)\Bigg|_{\lambda=\lambda_0} = 0\,,$$

which is equivalent to

$$\int_0^1 \mathrm{AVaR}_y(X)\left[1 + (1-\lambda_0)\log(y)\right] y^{-\lambda_0}dy + \alpha\beta q(1-q\lambda_0)^{-\beta-1} = 0\,.$$

Since $l_\lambda \in \hat{\mathcal{D}}^q_{\mathrm{s}}$ for all $\lambda \in [0,1/q)$, it follows from Theorem 6.13 that ρ is strictly monotone on L^p. However, it is possible that there exist $X \not\sim_{\mathrm{t}} Y$ in L^p for which the maximum in (8.15) is attained at the same $\lambda_0 \in [0,1/q)$. This means that $\hat{\chi}_\rho(q^X) \cap \hat{\chi}_\rho(q^Y) \neq \emptyset$. Hence, by Theorem 6.14, ρ is in general not strictly convex modulo translation.

References

Acerbi, C. (2002). *Spectral measures of risk: a coherent representation of subjective risk aversion.* J. Banking Fin. 26, 1505–1526.

Acerbi, C. (2004). *Coherent representation of subjective risk aversion.* In: Szegö, G. (ed.) Risk Measures For The 21st Century. New York: Wiley.

Artzner, P., Delbaen, F., Eber, J.M., Heath, D. (1999). *Coherent measures of risk.* Math. Finance 9(3), 203–228.

Ben-Tal, A., Teboulle, M. (1987). *Penalty functions and duality in stochastic programming via Φ-divergence functionals.* Math. Oper. Research 12, 224–240.

Cheridito, P., Delbaen F., Kupper M. (2006). *Dynamic monetary risk measures for bounded discrete-time processes.* Electronic Journal of Probability 11, 57–106.

Cheridito, P., Li, T. (2007). *Risk measures on Orlicz hearts.* To appear in Math. Finance.

Cherny, A. (2006). *Weighted V@R and its properties.* Fin. Stoch. 10(3), 367–393.

Cherny, A., Kupper, M. (2007). *Divergence utilities.* Preprint.

Csiszar, I. (1967). *On topological properties of f-divergences.* Studia Sci. Math. Hungarica 2, 329–339.

Dana, R. (2005). *A representation result for concave Schur concave functions.* Math. Finance 15(4), 613–634.

Delbaen, F. (2002). *Coherent risk measures on general probability spaces.* Advances in Finance and Stochastics, 39–56, Springer-Verlag, Berlin.

Delbaen, F. (2006). *Risk measures for non-integrable random variables.* Working Paper.

Edgar, G.A., Sucheston, L. (1992). Stopping Times and Directed Processes. Cambridge University Press.

Filipović, D., Kupper, M. (2007). *Monotone and cash-invariant convex functions and hulls.* Insurance: Mathematics and Economics 41(1), 1–16.

Filipović, D., Svindland, G. (2008). *Convex risk measures beyond bounded risks, or the canonical model space for law-invariant convex risk measures is L^1.* Working Paper.

Föllmer, H., Schied, A. (2002a). *Convex measures of risk and trading constraints.* Fin. Stoch. 6(4), 429–447.

Föllmer, H., Schied, A. (2002b). *Robust preferences and convex measures of risk.* Advances in Finance and Stochastics, 39–56, Springer-Verlag, Berlin.

Föllmer, H. and Schied, A. (2004). Stochastic Finance: An Introduction in Discrete Time. Second Edition. Walter de Gruyter Inc.

Frittelli, M., Rosazza Gianin, E. (2002). *Putting order in risk measures.* J. Banking Fin. 26(7), 1473–1486.

Frittelli, M., Rosazza Gianin, E. (2004). *Dynamic convex risk measures.* Risk Measures for the 21st Century, Chapter 12, Wiley Finance.

Frittelli, M., Rosazza Gianin, E. (2005). *Law invariant convex risk measures.* Adv. Math. Econ. 7, 42–53.

Hardy, G.H., Littlewood, J.E., Pólya, G. (1988). Inequalities. Reprint of the 1952 edition. Cambridge University Press, Cambridge.

Jouini, E., Schachermayer, W., Touzi, N. (2006). *Law invariant risk measures have the Fatou property.* Advances in Mathematical Economics 9, 49–71.

Kusuoka, S. (2001). *On law invariant coherent risk measures.* Advances in Mathematical Economics 3, 83-95.

Leitner, J. (2005). *A short note on second order stochastic dominance preserving coherent risk measures.* Math. Finance 15(4), 649-651.

Müller, A., Stoyan, D. (2002). Comparison Methods for Stochastic Models and Risks. Wiley Series in Probability and Statistics. John Wiley & Sons, Chichester.

Rockafellar, R.T., Uryasev, S., Zabarankin, M. (2006). *Generalized deviations in risk analysis.* Fin. Stoch. 10(1), 51–74.

Ruszczyński, A., Shapiro, A. (2006). *Optimization of convex risk functions.* Math. Op. Res. 31(3), 433–452.

Schied, A. (2007). *Optimal investments for risk- and ambiguity-averse preferences: A duality approach.* Fin. Stoch. 11(1), 107–129.

Shaked, M., Shanthikumar, J.G. (2007). Stochastic Orders. Springer.

Zălinescu, C. (2002). Convex Analysis in General Vector Spaces. World Scientific.