Representations of Gaussian measures that are equivalent to Wiener measure

Patrick Cheridito

Departement für Mathematik, ETHZ, 8092 Zürich, Switzerland.
E-mail: dito@math.ethz.ch

Summary. We summarize results on the representation of Gaussian measures that are equivalent to Wiener measure and discuss consequences for the law of the sum of a Brownian motion and an independent fractional Brownian motion.

Introduction

Let $0 < T \leq \infty$. For $0 < T < \infty$, we set $I_T = [0, T]$, and $I_\infty = [0, \infty)$. By $C(I_T)$ we denote the space of real-valued, continuous functions on I_T. The coordinates process $(X_t)_{t \in I_T}$ on $C(I_T)$ is given by

$$X_t(\omega) = \omega(t), \qquad \omega \in C(I_T),\ t \in I_T.$$

It generates the σ-algebra

$$\mathcal{B}_T := \sigma\big\{X_t^{-1}(B) : t \in I_T,\ B \text{ an open subset of } \mathbb{R}\big\}.$$

By $\mathbb{W}$ we denote Wiener measure on $(C(I_T), \mathcal{B}_T)$. We call a probability measure $\mathbb{Q}$ on $(C(I_T), \mathcal{B}_T)$ a Gaussian measure if $(X_t)_{t \in I_T}$ is a Gaussian process with respect to $\mathbb{Q}$. Such a measure is determined by its mean

$$M_t^{\mathbb{Q}} := \mathbb{E}_{\mathbb{Q}}[X_t], \qquad t \in I_T,$$

and its covariance

$$\Gamma_{ts}^{\mathbb{Q}} := \mathbb{E}_{\mathbb{Q}}\big[\big(X_t - M_t^{\mathbb{Q}}\big)\big(X_s - M_s^{\mathbb{Q}}\big)\big], \qquad t,\, s \in I_T.$$

We need some properties of integral operators induced by L^2-kernels. The proofs of the following facts can be found in Smithies (1958).

We denote by $L^2(I_T)$ and $L^2(I_T^2)$ the Hilbert spaces of equivalence classes of real-valued, square-integrable functions on I_T and I_T^2, respectively. An L^2-kernel is an element $f \in L^2(I_T^2)$. It induces a Hilbert–Schmidt operator

$$F : L^2(I_T) \longrightarrow L^2(I_T),$$

given by

$$Fa(t) = \int_0^T f(t,s)\,a(s)\,\mathrm{d}s, \qquad t \in I_T,\ a \in L^2(I_T).$$

The spectrum Spec(F) consists of at most countably many points. Every non-zero value in Spec(F) is an eigenvalue of finite multiplicity. If $(\lambda_j)_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, is the family of non-zero eigenvalues of F, repeated according to their multiplicity, then

$$\sum_{j=1}^N |\lambda_j|^2 < \infty.$$

Let $f, g \in L^2(I_T^2)$ with corresponding Hilbert–Schmidt operators F and G, respectively. Then, the kernel

$$\int_0^T f(t,u)\,g(u,s)\,\mathrm{d}u, \qquad t,\, s \in I_T,$$

is again in $L^2(I_T^2)$ and induces the operator product FG.

If $f \in L^2(I_T^2)$ and $1 \notin$ Spec(F), then there exists a unique kernel $\tilde{f} \in L^2(I_T^2)$ such that the corresponding operator $\tilde{F}$ satisfies

$$\mathrm{Id} - \widetilde{F} = (\mathrm{Id} - F)^{-1}.$$

Since $-\tilde{f}$ is usually called the resolvent kernel of f for the value 1, we call $\tilde{f}$ the negative resolvent kernel of f. It is the unique L^2-kernel $\tilde{f}$ that solves the equation

$$\tilde{f}(t,s) + f(t,s) = \int_0^T f(t,u)\,\tilde{f}(u,s)\,\mathrm{d}u, \qquad t,\, s \in I_T. \tag{1}$$

It is also the unique L^2-kernel that solves the equation

$$\tilde{f}(t,s) + f(t,s) = \int_0^T \tilde{f}(t,u)\,f(u,s)\,\mathrm{d}u, \qquad t,\, s \in I_T.$$

If f is symmetric, F is self-adjoint. Therefore, all eigenvalues λ_j are real, there exists a sequence $(e_j)_{j=1}^N$ of orthonormal eigenfunctions in $L^2(I_T)$, and f can be represented as

$$f(t,s) = \sum_{j=1}^N \lambda_j\, e_j(t)\, e_j(s),$$

where the series converges in $L^2(I_T^2)$. It follows that if f is symmetric and $1 \notin$ Spec(F), then

$$\tilde{f}(t,s) = \sum_{j=1}^{N} \frac{-\lambda_j}{1-\lambda_j} \, e_j(t) \, e_j(s). \tag{2}$$

In particular, $\tilde{f}$ is again symmetric. We set

$$S_T^1 := \big\{ f \in L^2\big(I_T^2\big) : f \text{ is symmetric and } \operatorname{Spec}(F) \subset (-\infty, 1) \big\}.$$

It can be seen from (2) that if $f \in S_T^1$, then $\tilde{f} \in S_T^1$ as well.

A kernel $g \in L^2(I_T^2)$ is called a Volterra kernel if $g(t,s) = 0$ for all $s > t$. In this case the corresponding operator G is quasi-nilpotent, that is, the spectral radius

$$\sup\{|\lambda| : \lambda \in \operatorname{Spec}(G)\} = \liminf_{n\to\infty} \|G^n\|^{1/n}$$

is zero. Hence, the negative resolvent kernel $\tilde{g}$ exists, and it can be shown that $\tilde{g}$ is also a Volterra kernel. We set

$$V_T := \big\{ g \in L^2\big(I_T^2\big) : g \text{ is a Volterra kernel} \big\}.$$

1 The representations of Shepp and Hitsuda

In the following theorem we recapitulate the statements of Theorems 1 and 3 of Shepp (1966). For $a, b \in L^2(I_T)$, we set

$$\langle a, b \rangle := \int_0^T a(s)\, b(s) \, \mathrm{d}s.$$

Theorem 1 (Shepp). (i) *Let $f \in S_T^1$ and $a \in L^2(I_T)$. Then*

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left(\int_0^T \int_0^s f(s,u) \, \mathrm{d}X_u \, \mathrm{d}X_s + \int_0^T a(s) \, \mathrm{d}X_s\right)\right] = \frac{\exp\Big(\frac{1}{2}\big\langle a, \big(\mathrm{Id} - \widetilde{F}\big)a\big\rangle\Big)}{\prod_{j=1}^N \exp(\lambda_j/2)\sqrt{1-\lambda_j}} < \infty,$$

where $(\lambda_j)_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, are the non-zero eigenvalues of the Hilbert–Schmidt operator induced by f and $\widetilde{F}$ is the Hilbert–Schmidt operator corresponding to the negative resolvent kernel $\tilde{f}$ of f. Furthermore, the probability measure

$$\mathbb{Q} = \frac{\prod_{j=1}^N \exp(\lambda_j/2)\sqrt{1-\lambda_j}}{\exp\Big(\frac{1}{2}\big\langle a, \big(\mathrm{Id} - \tilde{F}\big)a\big\rangle\Big)} \times \exp\left(\int_0^T \int_0^s f(s,u) \, \mathrm{d}X_u \, \mathrm{d}X_s + \int_0^T a(s) \, \mathrm{d}X_s\right) \cdot \mathbb{W} \tag{3}$$

is a Gaussian measure on $(C(I_T), \mathcal{B}_T)$ *with mean*

$$M_t^{\mathbb{Q}} = \int_0^t \big(\mathrm{Id} - \widetilde{F}\big) a(s) \, \mathrm{d}s, \qquad t \in I_T, \tag{4}$$

and covariance

$$\Gamma_{ts}^{\mathbb{Q}} = t \wedge s - \int_0^t \int_0^s \tilde{f}(u,v) \, \mathrm{d}v \, \mathrm{d}u, \qquad t,\, s \in I_T.$$

(ii) *Let* $\mathbb{Q}$ *be a Gaussian measure on* $(C(I_T), \mathcal{B}_T)$ *that is equivalent to* $\mathbb{W}$*. Then there exist unique* $f \in S_T^1$ *and* $a \in L^2(I_T)$ *such that* $\mathbb{Q}$ *has the representation (3).*

Remark 1. a) We call (3) the Shepp representation of the Gaussian measure $\mathbb{Q}$.

b) Let $k \in L^2(I_T^2)$ be symmetric. Then, k can be written as

$$k(t,s) = L^2\text{-}\lim_{n \to \infty} \sum_{|\lambda_j| \geq 1/n} \lambda_j \, e_j(t) \, e_j(s),$$

where $N \in \mathbb{N} \cup \{\infty\}$, $(\lambda_j)_{j=1}^N$ is a sequence of real numbers such that $\sum_{j=1}^N \lambda_j^2 < \infty$, and the e_j's are orthonormal in $L^2(I_T)$. Hence, under $\mathbb{W}$,

$$\begin{aligned} \int_0^T \int_0^s k(s,u) \, \mathrm{d}X_u \, \mathrm{d}X_s &= L^2\text{-}\lim_{n \to \infty} \sum_{|\lambda_j| \geq 1/n} \lambda_j \int_0^T \int_0^s e_j(s) \, e_j(u) \, \mathrm{d}X_u \, \mathrm{d}X_s \\ &= L^2\text{-}\lim_{n \to \infty} \sum_{|\lambda_j| \geq 1/n} \frac{\lambda_j}{2} \left(\left(\int_0^T e_j(s) \, \mathrm{d}X_s \right)^2 - 1 \right). \end{aligned}$$

Since the random variables $\int_0^T e_j(s) \, \mathrm{d}X_s$ are independent standard normal, it follows that

$$\mathbb{E}_{\mathbb{W}} \left[\exp\left(\int_0^T \int_0^s k(s,u) \, \mathrm{d}X_u \, \mathrm{d}X_s \right) \right] < \infty$$

if and only if $k \in S_T^1$.

c) Let $k \in L^2(I_T^2)$ be symmetric with corresponding Hilbert–Schmidt operator K. Then the map

$$\Gamma_{ts} = t \wedge s - \int_0^t \int_0^s k(u,v) \, \mathrm{d}v \, \mathrm{d}u, \qquad t,\, s \in I_T, \tag{5}$$

is the covariance function of a Gaussian process if and only if it is positive semi-definite, that is,

$$\left\langle \sum_{j=1}^n c_j \, \mathbf{1}_{[0,t_j]}, \; (\mathrm{Id} - K) \sum_{j=1}^n c_j \, \mathbf{1}_{[0,t_j]} \right\rangle = \sum_{j,l=1}^n c_j \Gamma_{t_j t_l} c_l \geq 0, \tag{6}$$

for all $n \in \mathbb{N}$, $\{t_1, \dots, t_n\} \subset I_T$ and $c \in \mathbb{R}^n$. Since the functions of the form

$$\sum_{j=1}^{n} c_j \, \mathbf{1}_{[0,t_j]}, \qquad n \in \mathbb{N},\ \{t_1, \dots, t_n\} \subset I_T,\ c \in \mathbb{R}^n,$$

are dense in $L^2(I_T)$, condition (6) is equivalent to

$$\langle c, (\mathrm{Id} - K)c \rangle \geq 0 \qquad \text{for all } c \in L^2(I_T).$$

Hence, (5) is the covariance function of a Gaussian process if and only if $\mathrm{Spec}(K) \subset (-\infty, 1]$.

Corollary 1. (i) *Let $(B_t)_{t \in I_T}$ be a Brownian motion and $(Z_t)_{t \in I_T}$ an independent Gaussian process. Then the law of $(B_t + Z_t)_{t \in I_T}$ is equivalent to $\mathbb{W}$ if and only if there exist $m \in L^2(I_T)$ and $k \in L^2(I_T^2)$ such that*

$$\mathbb{E}[Z_t] = \int_0^t m(s)\,\mathrm{d}s, \qquad t \in I_T, \tag{7}$$

and

$$\mathrm{Cov}(Z_t, Z_s) = \int_0^t \int_0^s k(u,v)\,\mathrm{d}v\,\mathrm{d}u, \qquad t, s \in I_T. \tag{8}$$

(ii) *Let $\mathbb{Q}$ be a Gaussian measure on $(C(I_T), \mathcal{B}_T)$ that is equivalent to $\mathbb{W}$ and $f \in S_T^1$ the kernel that satisfies (3). Then $\mathbb{Q}$ is the law of the sum of a Brownian motion and an independent Gaussian process if and only if* $\mathrm{Spec}(\widetilde{F}) \subset (-\infty, 0]$.

Proof. (i) Denote the law of $(B_t + Z_t)_{t \in I_T}$ by $\mathbb{Q}$. Then,

$$M_t^{\mathbb{Q}} = \mathbb{E}[B_t + Z_t] = \mathbb{E}[Z_t], \qquad t \in I_T,$$

and

$$\Gamma_{ts}^{\mathbb{Q}} = \mathrm{Cov}(B_t + Z_t, B_s + Z_s) = t \wedge s + \mathrm{Cov}(Z_t, Z_s), \qquad t,\ s \in I_T.$$

It follows from Theorem 1 that $\mathbb{Q}$ is equivalent to $\mathbb{W}$ if and only if (7) and (8) hold and $\mathrm{Spec}(K) \subset (-1, \infty)$. If (8) holds, then it can be shown as in Remark 1.c that $\mathrm{Spec}(K) \subset [0, \infty)$. This proves (i).

(ii) If $\mathbb{Q}$ is the law of the sum of a Brownian motion and an independent Gaussian process, then it follows as in Remark 1.c that $\mathrm{Spec}(\widetilde{F}) \subset (-\infty, 0]$.

If $\mathrm{Spec}(\widetilde{F}) \subset (-\infty, 0]$, then the function

$$-\int_0^t \int_0^s \tilde{f}(u,v)\,\mathrm{d}u\,\mathrm{d}v, \qquad t,\ s \in I_T, \tag{9}$$

is symmetric and positive semidefinite. Hence, there exists a Gaussian process $(Z_t)_{t \in I_T}$ with covariance (9) and mean $M^{\mathbb{Q}}$. Let $(B_t)_{t \in I_T}$ be an independent Brownian motion. Then, $\mathbb{Q}$ is the law of $(B_t + Z_t)_{t \in I_T}$. □

Example 1. For fixed $T \in (0,\infty)$ and $\alpha \in \mathbb{R} \setminus \{0\}$, let $\mathbb{Q}^{\alpha}_{H}$ be the law of the Gaussian process

$$B_t + \alpha B^H_t, \qquad t \in [0,T],$$

where $(B_t)_{t\in[0,T]}$ is a Brownian motion and $(B^H_t)_{t\in[0,T]}$ an independent fractional Brownian motion with Hurst parameter $H \in (0,1]$, that is, $(B^H_t)_{t\in[0,T]}$ is a Gaussian process with mean 0 and covariance

$$\operatorname{Cov}\big(B^H_t, B^H_s\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,\, s \in [0,T].$$

Since for $H \in (1/2,1]$,

$$\frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) = H(2H-1)\int_0^t\int_0^s |u-v|^{2H-2}\,\mathrm{d}v\,\mathrm{d}u, \quad t,\, s \in [0,T]\,,$$

it follows from Corollary 1.(i) that $\mathbb{Q}^{\alpha}_{H}$ is equivalent to $\mathbb{W}$ if and only if $H \in (3/4,1]$. This assertion is part of Theorem 1.7 in Cheridito (2001b), which was proved differently.

For $H \in (3/4,1]$, let λ_H be the largest eigenvalue of the operator K_H corresponding to the L^2-kernel

$$k_H(t,s) = H(2H-1)|t-s|^{2H-2}, \qquad t,\, s \in [0,T].$$

Since K_H is positive semi-definite, λ_H is equal to the operator norm $\|K_H\| > 0$ of K_H. If $\beta \in (0, 1/\lambda_H)$, then

$$t \wedge s - \frac{\beta}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,\, s \in [0,T],$$

is the covariance function of a centred Gaussian process equivalent to Brownian motion which cannot have the same law as the sum of a Brownian motion and an independent Gaussian process.

$$t \wedge s - \frac{1}{2\lambda_H}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,\, s \in [0,T],$$

is the covariance function of a centred Gaussian process that is neither equivalent to Brownian motion nor equal in distribution to the sum of a Brownian motion and an independent Gaussian process.

In the following theorem we reformulate the statements of Theorems 1' and 2' of Hitsuda (1968) (note that in the last line of Theorem 2' in Hitsuda (1968) X_t should be replaced by Y_t).

Theorem 2 (Hitsuda). (i) *Let $g \in V_T$ and $b \in L^2(I_T)$. Then*

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left(\int_0^T \left(\int_0^s g(s,u)\,\mathrm{d}X_u + b(s)\right)\mathrm{d}X_s \right.\right.$$
$$\left.\left. - \frac{1}{2}\int_0^T \left(\int_0^s g(s,u)\,\mathrm{d}X_u + b(s)\right)^2 \mathrm{d}s\right)\right] = 1,$$

and the probability measure

$$\mathbb{Q} = \exp\biggl(\int_0^T \Bigl(\int_0^s g(s,u)\,\mathrm{d}X_u + b(s)\Bigr)\,\mathrm{d}X_s - \frac{1}{2}\int_0^T \Bigl(\int_0^s g(s,u)\,\mathrm{d}X_u + b(s)\Bigr)^2 \mathrm{d}s\biggr)\cdot \mathbb{W} \quad (10)$$

is a Gaussian measure on $(C(I_T), \mathcal{B}_T)$*. Furthermore, the process*

$$B_t = X_t - \int_0^t \Bigl(\int_0^s g(s,u)\,\mathrm{d}X_u + b(s)\Bigr)\,\mathrm{d}s, \qquad t \in I_T, \quad (11)$$

is a Brownian motion with respect to $\mathbb{Q}$*, and*

$$X_t = B_t - \int_0^t\int_0^s \tilde{g}(s,u)\,\mathrm{d}B_u\,\mathrm{d}s + \int_0^t \bigl(\mathrm{Id} - \widetilde{G}\bigr)b(s)\,\mathrm{d}s, \qquad t \in I_T, \quad (12)$$

where $\tilde{g}$ *is the negative resolvent kernel of* g *and* $\widetilde{G}$ *the corresponding Hilbert–Schmidt operator.*

(ii) *Let* $\mathbb{Q}$ *be a Gaussian measure on* $(C(I_T), \mathcal{B}_T)$ *that is equivalent to* $\mathbb{W}$*. Then there exist unique* $g \in V_T$ *and* $b \in L^2(I_T)$ *such that* $\mathbb{Q}$ *has the representation (10).*

Remark 2. a) We call (10) the Hitsuda representation of the Gaussian measure $\mathbb{Q}$.

b) It follows from (11) and (12) that

$$\mathcal{F}_t^B = \sigma\{B_s : 0 \le s \le t\} = \sigma\{X_s : 0 \le s \le t\} = \mathcal{F}_t^X, \qquad t \in I_T.$$

Therefore, (12) is the canonical semimartingale decomposition of X in its own filtration. We call it the Hitsuda representation of the Gaussian process $((X_t)_{t\in I_T}, \mathbb{Q})$.

2 Relations between the representations of Shepp and Hitsuda

Theorem 3. *Let* $\mathbb{Q}$ *be a Gaussian measure on* $(C(I_T), \mathcal{B}_T)$ *that is equivalent to* $\mathbb{W}$ *and* f, $\tilde{f}$, g, $\tilde{g}$, a, b *the corresponding objects from Theorems 1 and 2. Then the following relations hold:*

$$\bigl(\mathrm{Id} - \widetilde{F}\bigr)a = \bigl(\mathrm{Id} - \widetilde{G}\bigr)b; \quad (13)$$

$$\mathbb{E}_{\mathbb{W}}\Bigl[\exp\Bigl(\int_0^T\int_0^s f(s,u)\,\mathrm{d}X_u\,\mathrm{d}X_s\Bigr)\Bigr] = \exp\Bigl(\frac{1}{2}\int_0^T\int_0^s g(s,u)\,\mathrm{d}u\,\mathrm{d}s\Bigr); \quad (14)$$

$$f(t,s) = g(t,s) - \int_t^T g(u,t)\,g(u,s)\,\mathrm{d}u, \qquad 0 \le s \le t \le T; \quad (15)$$

$$\tilde{f}(t,s) = \tilde{g}(t,s) - \int_0^s \tilde{g}(t,u)\,\tilde{g}(s,u)\,\mathrm{d}u, \qquad 0 \le s \le t \le T; \quad (16)$$

$$f(t,s) + \tilde{g}(t,s) = \int_s^T f(t,u)\,\tilde{g}(u,s)\,\mathrm{d}u, \qquad 0 \le s \le t \le T; \tag{17}$$

$$\tilde{f}(t,s) + g(t,s) = \int_0^t g(t,u)\,\tilde{f}(u,s)\,\mathrm{d}u, \qquad 0 \le s \le t \le T. \tag{18}$$

Proof. Relation (13) follows by comparing (4) and (12).

To prove the other relations we let $\mathbb{Q}_0$ be the Gaussian measure on $(C(I_T), \mathcal{B}_T)$ with mean 0 and the same covariance as $\mathbb{Q}$. It follows from Theorems 1 and 2 that

$$\mathbb{Q}_0 = \prod_{j=1}^N \exp\Big(\frac{\lambda_j}{2}\Big)\sqrt{1-\lambda_j}\exp\Big(\int_0^T\int_0^s f(s,u)\,\mathrm{d}X_u\,\mathrm{d}X_s\Big) \cdot \mathbb{W}$$

and

$$\mathbb{Q}_0 = \exp\Bigg(\int_0^T\int_0^s g(s,u)\,\mathrm{d}X_u\,\mathrm{d}X_s - \frac{1}{2}\int_0^T\Big(\int_0^s g(s,u)\,\mathrm{d}X_u\Big)^2\mathrm{d}s\Bigg) \cdot \mathbb{W}.$$

Now, relation (16) follows from the “only if” part of the proof of Proposition 2 in Hitsuda (1968) (note that in the corresponding equation in Hitsuda (1968) a variable u should be replaced by v). The relations (14), (15), (18) are equivalent to the equations (31d), (38), (10) in Kailath (1970), respectively (in equation (38) of Kailath (1970) there is a wrong sign).

Let $\widetilde{G}^*$ denote the adjoint of $\widetilde{G}$ and $(\mathrm{Id}-G)^*$ the adjoint of $(\mathrm{Id}-G)$. Then, relation (17) can be deduced from relation (16) as follows:

$$\begin{aligned}
(16) &\iff \big(\mathrm{Id}-\widetilde{F}\big) = \big(\mathrm{Id}-\widetilde{G}\big)\big(\mathrm{Id}-\widetilde{G}^*\big)\\
&\iff \mathrm{Id} = \big(\mathrm{Id}-F\big)\big(\mathrm{Id}-\widetilde{G}\big)\big(\mathrm{Id}-\widetilde{G}^*\big)\\
&\iff \big(\mathrm{Id}-G\big)^* = \big(\mathrm{Id}-F\big)\big(\mathrm{Id}-\widetilde{G}\big)\\
&\iff (17).
\end{aligned}$$

□

Remark 3. a) Relation (15) is equivalent to $(\mathrm{Id}-F) = (\mathrm{Id}-G^*)(\mathrm{Id}-G)$. Relation (18) is equivalent to $(\mathrm{Id}-\widetilde{G})^* = (\mathrm{Id}-G)(\mathrm{Id}-\widetilde{F})$.

b) In all four equations (15–18), either kernel is uniquely determined by the other.

Example 1 continued. Let $T \in (0,\infty)$, $H \in (3/4, 1]$, $\alpha \in \mathbb{R}\setminus\{0\}$ and $\mathbb{Q}^\alpha_H$ be the mean zero Gaussian measure from Example 1.

$$\Gamma^{\mathbb{Q}^\alpha_H}_{ts} = t \wedge s + \frac{\alpha^2}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \qquad t,\, s \in [0,T].$$

Hence,

$$\tilde{f}(t,s) = -\alpha^2 H(2H-1)|t-s|^{2H-2}, \qquad t,\, s \in [0,T]. \tag{19}$$

For the special case $H = 1$, (19) reduces to

$$\tilde{f}(t,s) = -\alpha^2, \qquad t,\, s \in [0,T],$$

and the equations (1), (18) and (16) can easily be solved. One obtains:

$$f(t,s) = \frac{\alpha^2}{1+\alpha^2 T}\,, \quad g(t,s) = \frac{\alpha^2}{1+\alpha^2 t}\,, \quad \tilde{g}(t,s) = -\frac{\alpha^2}{1+\alpha^2 s}\,, \quad t,\, s \in [0,T].$$

If $H \in (3/4, 1)$, it is less obvious how to find explicit expressions for the functions f, g, and $\tilde{g}$. The equations (1), (18) and (16) take the forms

$$\begin{aligned} f(t,s) + \alpha^2 H(2H-1) \int_0^T & f(t,u)|u-s|^{2H-2}\,\mathrm{d}u \\ &= \alpha^2 H(2H-1)|t-s|^{2H-2}, \qquad t,\, s \in [0,T], \end{aligned} \tag{20}$$

$$\begin{aligned} g(t,s) + \alpha^2 H(2H-1) \int_0^t & g(t,u)|u-s|^{2H-2}\,\mathrm{d}u \\ &= \alpha^2 H(2H-1)|t-s|^{2H-2}, \qquad t,\, s \in [0,T], \end{aligned} \tag{21}$$

and

$$\begin{aligned} \alpha^2 H(2H-1)&|t-s|^{2H-2} \\ &= -\tilde{g}(t,s) + \int_0^s \tilde{g}(t,u)\,\tilde{g}(s,u)\,\mathrm{d}u, \qquad 0 \le s \le t \le T, \end{aligned} \tag{22}$$

respectively. For certain values of H and α, equations (20) and (22) are solved in Sections 4.7 and 4.8 of Cheridito (2001a). Equation (21) can be solved similarly.

References

1. Cheridito, P. (2001a). *Regularizing Fractional Brownian Motion with a View towards Stock Price Modelling.* doctoral dissertation. `http://www.math.ethz.ch/~dito`
2. Cheridito, P. (2001b). Mixed fractional Brownian motion. *Bernoulli* 7(6), p. 913–934.
3. Hitsuda, M. (1968). Representation of Gaussian processes equivalent to Wiener process. *Osaka Journal of Mathematics* 5, p. 299–312.
4. Kailath, T. (1970). Likelihood ratios for Gaussian processes. *IEEE Transactions on Information Theory* IT-16, No. 3, p. 276–288.
5. Shepp, L.A. (1966). Radon–Nikodym derivatives of Gaussian measures. *Ann. Math. Statist.* 37, p. 321–354.
6. Smithies, F. (1958). *Integral Equations.* Cambridge Univ. Press, London and New York.